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The Graeffe Process as Applied to Power Series

Of the many methods which have been proposed for solving algebraic equations the most practical one, where complex roots are concerned, is the well known "root-squaring" method usually referred to as the GRAEFFE[1] process. Nearly all accounts of the method show by examples how all the roots of an equation may be found by first finding all the absolute values of the roots, and later by various devices their arguments. The problem of finding the absolute values of the roots is comparatively simple and can be extended (with certain practical difficulties) to the case of the zeros of entire functions. This step was made early in the history of the problem by EULER[2] in generalizing Bernoulli's method to determine the first few zeros of the Bessel function $J_0(z)$. More recent accounts of the problem of finding the absolute values of the zeros of entire functions have been given by OSTROWSKI[1] and PÓLYA.[3] The problem of finding by the Graeffe process the arguments of the zeros of entire functions (when these zeros are complex) apparently has not been attempted. At first this may seem a little strange since the Graeffe process owes its popularity to the fact that it is so successful in finding the complex roots of algebraic equations. The explanation is that the various devices, referred to above, for finding the arguments of the complex roots depend on an equation which has a finite number of roots.

There is however one modification of the Graeffe process for polynomials, given more than a score of years ago by BRODETSKY & SMEAL[4] and apparently overlooked by recent writers, in which a pair of complex roots may be found without having to find any other roots. The purpose of this note is to call attention to the fact that this process may be extended to entire functions and their complex zeros.

As mentioned above, accounts of the Graeffe process are nearly always given in terms of examples. To be sure, the "general ideas" of the process are also set forth, but I have yet to see a writer admit that the Graeffe process fails utterly in such simple cases as the equation

$$x^4 + x^3 + x^2 + x + 1 = 0$$

for the primitive fifth roots of unity. Although the Graeffe process is of considerable practical importance its theoretical justification, including the estimation of errors, is not at all easy. Even the lengthy memoir of Ostrowski, which is concerned largely with the absolute values of the roots, leaves something to be desired. If one is given only the coefficients of an equation with no advance information as to the disposition of its roots almost anything may happen if one applies the Graeffe process. Ostrowski shrewdly points out the distinction between the *a priori* and the *a posteriori* evaluation of the error, i.e. before and after the root-squaring process has been applied. It is clear that comparatively little of practical value may be said of the former. Even the number of steps necessary cannot be easily foreseen. The latter, unfortunately, depends on too many contingencies to afford a

simple treatment. In the brief discussion of the following method we must therefore confine ourselves to a few typical cases, leaving a more detailed and rigorous discussion to a later date and a more general method.

Let

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \qquad (a_i \text{ real}) \tag{1}$$

be an entire function whose zeros are required. We may suppose that $a_0 = 1$. We proceed to calculate for $m = 0, 1, 2, \cdots$ two sequences of numbers

$$\begin{array}{l} A_0^{(m)}, A_1^{(m)}, A_2^{(m)}, \cdots, A_r^{(m)}, \cdots \\ B_0^{(m)}, B_1^{(m)}, B_2^{(m)}, \cdots, B_r^{(m)}, \cdots \end{array} \tag{2}$$

defined as follows: For $m = 0$

$$A_r^{(0)} = a_r \qquad B_r^{(0)} = (r-1)a_{r-1}$$

so that $A_0^{(0)} = 1$, $B_0^{(0)} = B_1^{(0)} = 0$. For $m = 1$

$$A_r^{(1)} = (-1)^r a_r^2 + 2\sum_{\nu=0}^{r-1} (-1)^\nu a_\nu a_{2r-\nu}$$

$$B_r^{(1)} = \sum_{\nu=0}^{r-1} (-1)^\nu (2r - 2\nu - 1) a_\nu a_{2r-\nu-1},$$

and for $m > 1$

$$\begin{aligned} A_r^{(m)} &= (-1)^r (A_r^{(m-1)})^2 + 2\sum_{\nu=0}^{r-1} (-1)^\nu A_\nu^{(m-1)} A_{2r-\nu}^{(m-1)} \\ B_r^{(m)} &= \sum_{\nu=0}^{2r} (-1)^\nu A_\nu^{(m-1)} B_{2r-\nu}^{(m-1)}. \end{aligned} \tag{3}$$

For all m:

$$A_0^{(m)} = 1, \qquad B_0^{(m)} = 0. \tag{4}$$

Theoretically, the zeros of $f(z)$ are, in general, limits, as $m \to \infty$, of simple functions of $A_r^{(m)}$ and $B_r^{(m)}$. In what follows it will be convenient to let s denote 2^m. Let the reciprocals of the zeros of (1) be denoted by $\alpha_k (k = 1, 2, \cdots)$, the notation being such that

$$|\alpha_1| \geqslant |\alpha_2| \geqslant |\alpha_3| \geqslant \cdots. \tag{5}$$

Finally let ϵ be an independent variable. Then the function defined recurrently by

$$\varphi_0 = \varphi_0(t) = \varphi_0(t|\epsilon) \equiv \sum_{r=0}^{\infty} a_r (t - \epsilon)^{-r}$$

$$\varphi_m(t) = \varphi_{m-1}(\sqrt{t})\varphi_{m-1}(-\sqrt{t}),$$

has the expansion

$$\varphi_m(t) = \sum_{r=0}^{\infty} t^{-r}(A_r^{(m)} + \epsilon s B_r^{(m)}) + 0(\epsilon^2) \tag{6}$$

where $A_r^{(m)}$, $B_r^{(m)}$ are given by (3). Moreover the zeros of this function are at

$$t = \alpha_k^s + \epsilon s \alpha_k^{s-1}. \tag{7}$$

Identifying the elementary symmetric functions of (7) with the coefficients of (6) and recalling the definition of ϵ and (4), we have, in the notation of symmetric functions

$$\begin{array}{ll} A_1^{(m)} = -\sum \alpha_1^s & B_1^{(m)} = -\sum \alpha_1^{s-1} \\ A_2^{(m)} = \sum \alpha_1^s \alpha_2^s & B_2^{(m)} = \sum \alpha_1^s \alpha_2^{s-1} \\ A_3^{(m)} = -\sum \alpha_1^s \alpha_2^s \alpha_3^s & B_3^{(m)} = -\sum \alpha_1^s \alpha_2^s \alpha_3^{s-1} \\ \cdots\cdots\cdots & \cdots\cdots\cdots \end{array} \tag{8}$$

It is from these equations that we derive approximations to α_k. Different cases arise, depending on which of the $\geqslant$ signs in (5) may be replaced by $=$. The simplest case is that in which

$$|\alpha_1| > |\alpha_2| > |\alpha_3| > \cdots. \tag{9}$$

Although one might call this the "general" case, a moment's reflection shows that in this case all the zeros of $f(z)$ are real. This is therefore a special case. By the usual argument, the symmetric functions (8) are, in this case, dominated by their first terms, provided s is so large that $A_r^{(m)}$ is sensibly the square of $A_r^{(m-1)}$. Hence, approximately

$$\begin{array}{ll} A_1^{(m)} = -\alpha_1^s & B_1^{(m)} = -\alpha_1^{s-1} \\ A_2^{(m)} = \alpha_1^s \alpha_2^s & B_2^{(m)} = \alpha_1^s \alpha_2^{s-1} \\ A_3^{(m)} = -\alpha_1^s \alpha_2^s \alpha_3^s & B_3^{(m)} = -\alpha_1^s \alpha_2^s \alpha_3^{s-1} \\ \cdots\cdots\cdots & \cdots\cdots\cdots \end{array}$$

In general, equations of this type continue as long as $>$ signs continue to hold in (9). A knowledge of the A's alone suffices to determine $|\alpha_k|$ approximately by the usual relations

$$\begin{array}{l} |\alpha_1| = |A_1^{(m)}|^{1/s} \\ |\alpha_2| = |A_2^{(m)}/A_1^{(m)}|^{1/s} \\ |\alpha_3| = |A_3^{(m)}/A_2^{(m)}|^{1/s} \\ \cdots\cdots\cdots \end{array} \tag{10}$$

To determine the signs of these real roots it has been customary[5] to substitute $\pm\alpha_k$ into the equation $f(z) = 0$, or to draw a graph of $f(z)$. But knowledge of the B's provides a rational method of determining the values of α_k (with their signs). In fact, we have approximately

$$\begin{array}{l} \alpha_1 = A_1^{(m)}/B_1^{(m)} \\ \alpha_2 = A_2^{(m)}/B_2^{(m)} \\ \alpha_3 = A_3^{(m)}/B_3^{(m)} \\ \cdots\cdots \end{array} \tag{11}$$

The agreement of the two values of $|\alpha_k|$ obtained in (10) and (11) affords an excellent check on the work. If, however, the B's are calculated only roughly, the signs of α_k can be determined from the signs of $B_k^{(m)}$, since $s - 1$ is odd.

Suppose now that there are zeros of equal absolute value. The most important case of this kind is that in which the α's occur in pairs of complex conjugate values, no two pairs having the same absolute value. This has been called the "general case." To discuss it briefly we write

$$\alpha_1 = \rho_1 e^{i\theta_1},\ \alpha_2 = \rho_1 e^{-i\theta_1},\ \cdots,\ \alpha_{2k-1} = \rho_k e^{i\theta_k},\ \alpha_{2k} = \rho_k e^{-i\theta_k}$$

where the absolute values ρ_k satisfy

$$\rho_1 > \rho_2 > \rho_3 > \cdots.$$

For large s, (8) gives in this case, as dominant terms,

$$\begin{aligned}
A_1^{(m)} &= -\,2\rho_1^s \cos s\theta_1 \\
A_2^{(m)} &= \rho_1^{2s} \\
A_3^{(m)} &= -\,2\rho_1^{2s}\rho_2^s \cos s\theta_2 \\
A_4^{(m)} &= \rho_1^{2s}\rho_2^{2s} \\
B_1^{(m)} &= -\,2\rho_1^{s-1} \cos (s-1)\theta_1 \\
B_2^{(m)} &= 2\rho_1^{2s-1} \cos \theta_1 \\
B_3^{(m)} &= -\,(2\rho_1^{2s}\rho_2^{s-1} \cos (s-1)\theta_2 + 4\rho_1^{2s-1}\rho_2^s \cos \theta_1 \cos s\theta_2) \\
B_4^{(m)} &= 2\rho_1^{2s}\rho_2^{2s-1} \cos \theta_2 + 2\rho_1^{2s-1}\rho_2^{2s} \cos \theta_1.
\end{aligned} \tag{12}$$

This case is therefore recognized by the oscillatory character of $A_{2k+1}^{(m)}$ as functions of m, together with the fact that $A_{2k}^{(m)}$ approaches the square of $A_{2k}^{(m-1)}$ $(k = 0, 1, 2, \cdots)$. It is now clear how we may determine (ρ_1, θ_1), (ρ_2, θ_2), $\cdots$ and verify our results. In fact

$$\begin{gathered}
\rho_1 = |A_2^{(m)}|^{1/2s},\ \rho_2 = |A_4^{(m)}/A_2^{(m)}|^{1/2s},\ \cdots \\
2\cos\theta_1 = \rho_1 B_2^{(m)}/A_2^{(m)},\ 2\cos\theta_2 = \rho_2\left(\frac{B_4^{(m)}}{A_4^{(m)}} - \frac{B_2^{(m)}}{A_2^{(m)}}\right),\ \cdots.
\end{gathered} \tag{13}$$

The values of (ρ_1, θ_1), (ρ_2, θ_2), (ρ_3, θ_3), $\cdots$ thus found may be verified by using the expressions of $A_1^{(m)}, A_3^{(m)}, \cdots$ and $B_1^{(m)}, B_3^{(m)}, \cdots$. If the B's are calculated only roughly the values of $\theta_1, \theta_2, \cdots$ obtained from them will be sufficiently exact to determine which one of the *s a priori* values of θ_k as defined by

$$2\cos s\theta_k = \frac{-\,A_{2k-1}^{(m)}}{\sqrt{|A_{2k}^{(m)} A_{2k-2}^{(m)}|}}$$

is the correct θ_k to choose. The case of more than two zeros having equal absolute values breaks into subcases, each recognizable from the behavior of the A's and each capable of a similar though more complicated treatment.

Of course $f(x) = 0$ may have some real and some complex zeros. In practice this is especially true of polynomials but rather rare in the case of typical entire functions. In this case the above discussions are only slightly modified. Thus if the first three zeros α_1, α_2, α_3 are real and all the other α's are complex, equations (10) and (11) hold for α_1, α_2, and α_3, while equations (12) and (13) are modified by replacing $A_k^{(m)}$, $B_k^{(m)}$ by $A_{k+3}^{(m)}/A_3^{(m)}$, $B_{k+3}^{(m)}/A_3^{(m)}$ in accordance with the fundamental equations (8).

We consider now certain practical difficulties which arise when one extends the Graeffe process to entire functions. The most conspicuous of

these is due to the fact that the $A_r^{(m)}$ and $B_r^{(m)}$ are each functions of the first $2^m r$ coefficients of $f(z)$ so that even for moderate values of r and m a large number of these coefficients must be known. Of course if $f(z)$ is merely a polynomial of moderate degree the difficulty does not arise and most of the terms in the calculation of $A_r^{(m)}$ and $B_r^{(m)}$ vanish. To put it another way, the Graeffe process for a polynomial f of degree n makes considerable use of the fact that the coefficients of f, after the n-th, are exactly zero. This should warn the computer to be careful in truncating a power series if he expects the roots of the polynomial thus obtained to agree with the zeros of the series, especially for large zeros. In many typical cases the coefficients a_n for n large may be closely approximated not by zero but by some simple asymptotic formula. This fact will reflect itself in the formulas (3) for A and B, thus enabling one to estimate the contribution from the far distant coefficients. In many cases also the function $f(z)$ possesses a simple asymptotic expansion for large z which enables one to compute (with bounded accuracy) a large zero of $f(z)$. In these cases the Graeffe process may be used for finding only those zeros which are comparatively near the origin.

Another source of trouble is loss of accuracy in computing the A's and B's. This may be due to the accumulation of errors from "rounding off" the coefficients in the given equation. If enough accuracy can be maintained until the process begins to converge, that is, until the expressions for $A_r^{(m)}$ and $B_r^{(m)}$ in (3) are dominated by their first few terms, then the crisis is passed and no great loss of accuracy occurs thereafter. This difficulty increases when the roots are nearly equal and is intolerable (as Ostrowski points out) when all the roots coincide, a case which is however of no practical importance.

A more insidious source of trouble is a loss of accuracy due to cancellation of terms in computing $A_r^{(m)}$ and $B_r^{(m)}$. Here the extreme case is that in which $f(z) = \exp g(z)$, where g is an odd function. In this case one may show that $A_k^{(1)} = 0$ for $k > 0$. In particular if $f(z) = e^{cz}$ the B's also vanish with the exception of $B_1^{(1)} = 2c$. This puts an end to the Graeffe process. Of course in this case the function $f(z)$ has no zeros anyway. The converse proposition is also true: If $A_k^{(1)} = 0$ for $k > 0$, and $B_k^{(1)} = 0$ for $k > 1$, then $f(z) = e^{cz}$ and $c = \frac{1}{2}B_1^{(1)}$. Hence in practice we need not expect an outright failure of the process. However, if the coefficients differ but slightly from those of an exponential function one may expect a considerable cancellation of terms in computing $A_r^{(m)}$ and $B_r^{(m)}$, resulting in a devastating loss of accuracy. Hence extreme accuracy is required in the given coefficients a_r. If the a_r are defined as rational numbers it may be advisable to use these exact values rather than any approximate decimal equivalents. In this way the A's and B's appear as exact rational numbers, no accuracy being lost. This device may have to be used for several stages until the process shows signs of converging. If the a_r are given to within only a certain degree of accuracy and there is this tendency toward cancellation, very little can be done about it and some other method may have to be used. This difficulty is only slightly noticeable when $f(z)$ is a polynomial of low degree. It is quite pronounced in the following example.

Example. The following equation, proposed to me by F. Wolf, arises in

connection with the flow of a hot gas through a pipe:

$$J_1^2(z) = J_0(z)J_2(z)$$

where $J_n(z)$ is the ordinary Bessel function of the first kind. It may be shown that

$$z^2\Delta(z) \equiv z^2[J_1^2(z) - J_0(z)J_2(z)] = 4\sum_{n=2}^{\infty} nJ_n^2(z)$$

from which it follows that except for the trivial double zero at $z = 0$, $\Delta(z)$ has no real or imaginary zeros. Replacing z^2 by $-4x$ and expanding J_0, J_1 and J_2 in powers of x one finds that

$$f(x) \equiv \frac{-2\Delta}{x} = 1 + \frac{2}{3}x + \frac{5}{24}x^2 + \frac{7}{180}x^3 + \cdots$$
$$= 2\sum_{m=0}^{\infty}\binom{2m+2}{m}\frac{x^m}{(m+1)!(m+2)!}$$

If we wish to compute for example $A_{11}^{(1)}$ and we carry the calculations of the terms to ten significant figures we obtain (in units of the 33rd decimal place)

$$\begin{array}{rrr}
-a_{11}^2 & & = -28013\ 32771 \\
2a_{10} & a_{12} = & 48133\ 66627 \\
-2a_9 & a_{13} = & -30467\ 02393 \\
2a_8 & a_{14} = & 14129\ 74452 \\
-2a_7 & a_{15} = & -4754\ 97203 \\
2a_6 & a_{16} = & 1144\ 16515 \\
-2a_5 & a_{17} = & -192\ 74739 \\
2a_4 & a_{18} = & 22\ 06130 \\
-2a_3 & a_{19} = & -1\ 64295 \\
2a_2 & a_{20} = & 7456 \\
-2a_1 & a_{21} = & -185 \\
2a_0 & a_{22} = & 2
\end{array}$$

$$A_{11}^{(1)} = \sum_{r=0}^{22}(-1)^r a_r a_{22-r} = -404$$

Hence only three significant figures of $A_{11}^{(1)}$ are obtained from 10-figure data. If however the a's are taken at their rational values the rational value of $A_{11}^{(1)}$ is found to be exactly

$$-19471\ 33037/(2^{35}\cdot 3^{19}\cdot 5^8\cdot 7^5\cdot 11^4\cdot 13^2\cdot 17\cdot 19\cdot 23) =$$
$$-4.04020\ 96279\ 4\ \ldots\cdot 10^{-31}.$$

In this example the first 32 coefficients of $f(x)$ were used to find

$$(14)\quad \begin{array}{ll}
A_1^{(4)} = 1.9683\ 98486\cdot 10^{-12} & B_1^{(4)} = -1.4572\ 73376\cdot 10^{-11} \\
A_2^{(4)} = 1.7561\ 21066\cdot 10^{-24} & B_2^{(4)} = -1.5624\ 51484\cdot 10^{-23} \\
A_1^{(5)} = -3.62350470\cdot 10^{-25} = 2\rho_1^{32}\cos 32\theta_1 & \\
B_1^{(5)} = 1.3060\ 43223\cdot 10^{-23} = 2\rho_1^{31}\cos 31\theta_1 & \\
A_1^{(3)} = -8.2527\ 36680\cdot 10^{-7} & B_1^{(3)} = -3.3783\ 23209\cdot 10^{-6}
\end{array}$$

We first find that

$$[A_2^{(4)}]^{1/32} = 0.18098\ 48935$$
$$[A_2^{(4)}]^{1/32}B_2^{(4)}/A_2^{(4)} = -1.6102\ 54104.$$

These are approximations to ρ_1 and $2\cos\theta_1$ respectively. Substituting these into (14) we find by successive approximations that (14) has the simultaneous solution

$$\rho_1 = 0.1809\ 84894$$
$$\theta_1 = 2.5066\ 8639.$$

In this calculation it is not feasible to carry the main root-squaring process any further. One could have wished for more coefficients and higher powers, to separate more zeros. In this case the remaining zeros can be approximated by an asymptotic formula with sufficient accuracy except in the case of α_3 and α_4 which are still fairly close to the origin. The formula gives only

$$\rho_2 = 0.063, \quad \theta_2 = 2.70.$$

Since the larger zeros are well determined and since we now know the first pair of complex zeros, it is a simple matter to compute the sums of the kth powers of all the zeros except α_3 and α_4 for $k = 7$ and 8. From the above values of $A_1^{(3)}$ and $B_1^{(3)}$ it follows that

$$\alpha_3^8 + \alpha_4^8 = 2\rho_2^8 \cos 8\theta_2 = -5.4461 \cdot 10^{-10}$$
$$\alpha_3^7 + \alpha_4^7 = 2\rho_2^7 \cos 7\theta_2 = \ \ 9.1125 \cdot 10^{-9}$$

The approximate values of ρ_2 and θ_2 given above make it easy to determine by successive approximations the solution

$$\rho_2 = 0.064339, \quad \theta_2 = 2.70071$$

of the above system.

The above example is given in such detail in order to indicate what steps may be necessary in dealing with intractable entire functions. When one is deprived of the use of extremely high powers of the zeros (customarily used in treating polynomials) one is obliged to obtain more information than usual from the lower powers.

D. H. L.

[1] For a typical account of this method, together with some historical remarks, see E. T. WHITTAKER & G. ROBINSON, *The Calculus of Observations*, third ed., London, 1940, p. 106–120. References to other accounts are given in C. A. HUTCHINSON, "On Graeffe's method for the numerical solution of algebraic equations," *Am. Math. Mo.*, v. 42, 1935, p. 149–161, and L. L. CRONVICH, "On the Graeffe method of solution of equations," *Am. Math. Mo.*, v. 46, 1939, p. 185–190. An early exponent of this method, who did much to make it well known, was the astronomer J. F. ENCKE (*J. f. d. reine u. angew. Math.*, v. 22, 1841, p. 193f). He attributed the method to Graeffe. He was also responsible for an utterly useless scheme of changing the signs of all the roots. Many writers on the subject still use this confusing notion of the "Encke roots" of an equation. Graeffe's method of solutions of numerical equations was really due to DANDELIN (1794–1847), *Mem. Acad. Brussels*, v. 3, 1826, p. 46, or to WARING (1734–1798), *Meditationes Analyticae*, 1776, p. 311. It was published by C. H. GRAEFFE (1799–1873) in a prize paper, *Die Auflösung der höheren numerischen Gleichungen*, Zürich, 1837. Other accounts of the history of the method will be found in A. M. OSTROWSKI, "Recherches sur la méthode de Graeffe et les zéros des polynomes et des séries de Laurent," *Acta Math.*, v. 72, 1940, p. 99–155, and E. CARVALLO, *Méthode pratique pour la Résolution numérique complète des équations algébriques ou transcendantes*, Paris diss., 1890.

[2] L. EULER, Akad. Nauk, S.S.S.R., Leningrad, *Acta*, 1781, part 1, 1784, p. 170f.

[3] G. PÓLYA, "Über das Graeffesche Verfahren," *Z. Math. Phys.*, v. 63, 1915, p. 275–290.

[4] S. BRODETSKY & G. SMEAL, "On Graeffe's method for complex roots of algebraic equations," Camb. Phil. So., *Proc.*, v. 22, 1924, p. 83f.

[5] A. C. AITKEN has proposed a root-cubing procedure in which real roots are determined together with their signs; see *Math. Gazette*, v. 15, 1931, p. 490–491.

RECENT MATHEMATICAL TABLES

184[A, B, C, D, E, M].—EDWARD SCHWITZER ALLEN (1887–), *Six-Place Tables. A Selection of Tables of Squares, Cubes, Square Roots, Cube Roots, Fifth Roots and Powers, Circumferences and Areas of Circles, Common Logarithms of Numbers and of the Trigonometric Functions, the Natural Trigonometric Functions, Natural Logarithms, Exponential and Hyperbolic Functions, and Integrals.* Sixth ed., New York, McGraw-Hill, 1941. xxiii, 181 p. 10.7 × 17.7 cm. Leatherette $1.50. Compare RMT 49.

Through revisions, including enlargements, and the addition of marginal index and flexible covers, in new editions (1925, 1929, 1931, 1935, 1941) since the original 1922 pocket volume, this work has attained to its present very useful form. At many places of the compilation, the thorough competence of the author is in evidence. In this latest edition three new tables are added, namely: (a) T. VIII, a four-place table of sine, cosine, tangent, cotangent, and log sin, log cos, log tan, for radian argument [0(.01)3.20; 4D], included at the suggestion of electrical engineers; (b) T. IX, $\log n!$, $n = [1(1)100; 6D]$; and (c) a four-place table of complete elliptic integrals K, E, for $\sin^{-1} k = 0(1°)65°(0°.5)80°(0°.2)89°(0°.1)89°.9$ added to T. XIII, containing 167 integrals.

T. VI is a six-place table of all six of the natural trigonometric functions for each minute of the quadrant; T. XI, for e^x, e^{-x}, $\sinh x$, $\cosh x$, $\tanh x$, $x = [0(.01)1(.05)2.4(.1)5(1)10;$ mostly 4D], and $\Gamma(x)$, $x = [0(.01).99; 4D]$, and $\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, $x = [0(.01)2.99; 4D]$. Thus not all tables are six-place; T. III of circumferences and areas of circles with diameters from 1/64 to 100 is mostly 6S. The last Table, XIV, is devoted to mathematical constants. In the xxiii pages of the Introduction the theory of logarithms and explanation of tables are given, with exercises for solution.

R. C. A.

185[A, B, D, E, G, H, I].—FRITZ EMDE, *Tables of Elementary Functions . . . With 83 figures. Tafeln elementarer Funktionen . . . mit 83 Textfiguren.* Leipzig and Berlin, Teubner, 1940. Two title-pages. On the back of the second title-page is, in part, *Copyright vested in the Alien Property Custodian, 1944, pursuant to law. Published and distributed in the Public Interest by authority of the Alien Property Custodian under License No. A-709. Lithoprinted by Edwards Brothers, Inc., Ann Arbor, Michigan, U. S. A.*, 1945. xii, 181 p. 17.7 × 26.1 cm. $3.20.

There was incidental reference to this work in *MTAC*, p. 108, Oct. 1943 (when no copy of the work was available in this country), in the course of reviews of the six editions or reprints then available of JAHNKE & EMDE, *Tables of Functions*. The first 78 p., devoted to Elementary Functions in the third edition of 1938, did not appear in the fourth or fifth editions but were reprinted in the sixth. The work under review was intended to be a revision, and in various directions greatly to elaborate the 1938 material. Larger type is used and the type-page is about 25% larger than that of the *Tables of Functions* of JAHNKE & EMDE. Instead of X sections in the original 78 pages there are now XVII sections in the 172 pages. We shall now briefly indicate most of what is new. Section I, **Powers,** p. 1–19 (old, p. 1–11); the new tables are of squares and reciprocals, tables of cubes and reciprocals of the squares, powers of 2, 3, 5, 7, 11, $x^{\frac{1}{2}}$, $(10x)^{\frac{1}{2}}$, $(x/10)^{1/3}$, $x^{1/3}$, $(10x)^{1/3}$. Section II, **Factor Table, Factorials,** etc., p. 20–25 (new): Factor table of 49–9997; n!, 1/n!, n!!, 1/n!!, n = 1(1)20; multiples of $\log e = M$, x/M, $2x/\pi$, $\pi x/2$. Section III, **Auxiliary tables for computation with complex numbers,** p. 26–33 (old, p. 12–19), slight rearrangements. Section IV, **Quadratic equations,** p. 34–37 (new), graphs and tables for the solution of

$x^3 + px \mp q^3 = 0$. Section V, **Cubic equations,** p. 38–47 (old, p. 20–30), much the same. Section VI, **Equations of the fourth degree,** p. 47–72 (new), elaborate graphs, and tables for finding real and complex roots of biquadratic equations. Section VII, **Circular functions, angles in degrees,** p. 73–79 (new), tables of $\sin x$, $\cos x$, $\tan x$, $\sqrt{2}\sin x$, $\sqrt{2}\cos x$, $1 - \cos x$; also of integrals; graphs; etc. Section VIII, **Angles in quadrants,** p. 80–89 (new), tables of amp. $\frac{1}{2}\pi x$, $\sin x$, $\sinh \frac{1}{2}\pi x$, $e^{-\frac{1}{2}\pi x}$, $\cos x$, $\cosh \frac{1}{2}\pi x$, $e^{\frac{1}{2}\pi x}$, $\tan x$, $\tanh \frac{1}{2}\pi x$, $\coth \frac{1}{2}\pi x$, $\sqrt{2}\sin x$, $\sqrt{2}\cos x$, $1 - \cos x$, $\cosh \frac{1}{2}\pi x - 1$; also graphs. Section IX, **Hyperbolic functions, angles in radians,** p. 90–115 (old, p. 52–60), various formulae, functions of a multiple sector, powers, derivatives, integrals, approximate values for small sectors, gudermannian; graphs and altitude chart; numerous tables include those of $\sin x$, $\cos x$, $\tan x$, $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\sinh x$, $\cosh x$, $\tanh x$, $\sinh^{-1} x$, $\cosh^{-1} x$, e^x, e^{-x}, $1 - \cos x$, $\cosh x - 1$; tables of $\ln x$ and $\log x$. Section X, **Special functions,** p. 116–130 (old, p. 36–39, 44–51), tables of e^{-x^3}, Planck's radiation function, Langevin's function $\coth x - 1/x$, $\rho \tan \rho$, $-\cot \rho/\rho$, $-\tan \rho/\rho$, ρ in right angles; numerous graphs including Hayashi's $\sinh x \sin x$, $\cosh x \cos x$, $\sinh x \cos x$, $\cosh x \sin x$. Section XI, **Transcendental equations,** p. 130–131 (old, p. 30–31). Section XII, **Circular and hyperbolic functions of a complex variable,** p. 132–145 (old, p. 60–76), some revisions. Section XIII, **The function $[\tan (i^{\frac{1}{2}}r)]/i^{\frac{1}{2}}r$,** p. 146–155 (new), graphs and table of the function. Section XIV, **Chebyshev's polynomials,** p. 156–157 (new), graphs. Section XV, **The function z^z,** p. 158 (new), graph. Section XVI, **Approximate calculations with polynomials,** p. 159–162 (new), formulae, tables. Section XVII, **Some remarks on numerical calculations,** p. 163–172 (new). The tables on old pages 32–35 and 40–44 do not appear in the newer work.

On pages 173–176 is a list of nearly 100 "Useful books for the computer."

In various tables the interval in the argument is either 1 or 2 or 5 units in the last figure. The differences printed are calculated throughout for the interval 1. "The interval is chosen so great that the maximal error which might arise by linear interpolation does not exceed .5 (10^{-4}) of the function value. Further details on this point are to be found in the last chapter. This increase of the interval reduces the tables considerably, so that the values for many functions can be given in small space. The tables are not overstretched, but adapted to the demand of each function."

The work is in English and German throughout, the English translations having been prepared by G. W. O. Howe of the University of Glasgow.

Emde felt that his "book will be useful to many who never refer to the 1938 Jahnke-Emde. The latter also include most of the students of Technical Colleges. Our book may prove a help in their mathematical instruction. The numerous figures to scale, of which many show the same function in various scales, will make the functions clearer. An attentive study of the variation of the interval in the several functions will lead to greater insight."

R. C. A.

186[A, D].—H. M. Terrill & Ethel M. Terrill, "Tables of numbers related to the tangent coefficients," Franklin Institute, *J.*, v. 239, Jan. 1945, p. 66–67. 16.5 × 24.1 cm.

$$\text{If } \tan x = T_1 x + T_2 \frac{x^3}{3!} + T_3 \frac{x^5}{5!} + T_4 \frac{x^7}{7!} + \cdots,$$

$$T_n = 2^{2n}(2^{2n} - 1)B_n/2n.$$

Correcting the erroneous relation in the text, the number H_n should have been defined by $nH_n = 2^n(2^{2n} - 1)B_n$. Hence

$$H_n = T_n/2^{n-1}.$$

Defining the number $K_n = 2(2^{2n} - 1)B_n$, we have $K_n = nT_n/2^{2n-2}$, and

$$nH_n = 2^{n-1}K_n.$$

The chief tables before us give the values of H_n and K_n (each an integer) for $n = 1(1)19$. Since there is a table of T_n for $n = 1(1)30$, in Peters and Stein's *Anhang*, p. 88, of Peters' *Zehnstellige Logarithmentafel*, v. 1, Berlin, 1922, the computation of the present tables for $n = 1(1)19$ was a rather trivial matter. There are also two small tables of H_n^p and K_n^p, for $p = 1(1)4$, $n = 2(1)5$; these numbers are defined by the recurrence formulae

$$2p^2H_n^p - (p+1)(2p+1)H_n^{p+1} = H_{n+1}^p,$$
$$p^2K_n^p - (p+1)^2K_n^{p+1} = K_{n+1}^p,$$

where $H_1^p = K_1^p = 1$.

In his *Théorie des Nombres*, Paris, 1891, v. 1, p. 251, LUCAS refers to $G_{2n} = (-1)^n K_n$ as Genocchi numbers (A. GENOCCHI, *Annali d. Sci. Matem.*, v. 3, 1852, p. 395 f.), though considered by EULER (*Institutiones Calculi Differentialis*, v. 2, 1755, p. 486), and Lucas gives the first six numbers of the Terrill table. A ms. table of K_n for $n \leq 110$, by D. H. L., is in the Library of the American Mathematical Society.

R. C. A.

187[B, D, E, G, H, L].—P. R. E. JAHNKE & F. EMDE, *Tables of Functions with Formulae and Curves* [*Funktionentafeln mit Formeln und Kurven . . .*] *Published and distributed in the public interest by authority of the Alien Property Custodian under license No. A-57*, Fourth revised edition, New York, Dover Publications, 1780 Broadway, 1945. [xv], 306, 76 p. 15.4 × 29.8 cm. $3.75.

In *MTAC*, p. 106–109, 171 and 204 (corrigenda), 293–294, some information has been given regarding the five previous editions or reprints in 1909, 1933, 1938, 1941, and 1943. Now we have the sixth, with the correction of 396 errors in the fifth edition (or reprint), and the addition of a Supplementary Bibliography containing 43 titles. Material for correction of the errors was mainly furnished, either directly or indirectly, by R. C. A., L. J. C., NYMTP, and J. C. P. MILLER. In many cases these errors, on 56 pages, are only unit errors in the last decimal place. A list of the pages where the only known errors in the 1943 volume occur, followed by their number on each page, may be found to be of service. They are as follows:

9 (1), 10 (2), 34 (8), 35 (3), 39 (1), 59 (2), 62 (1), 65 (1), 66 (1), 69 (3), 70 (1), 72 (3), 74 (1), 79 (1), 85 (3), 94 (1), 102 (2), 103 (1), 105 (1), 117 (1), 124 (13), 125 (40), 128 (2), 164 (32), 166 (1), 168 (27), 177 (1), 183 (1), 204 (2), 205 (1), 206 (4), 212 (1), 216 (2), 218 (1), 220 (1), 228 (15), 229 (6), 234 (8), 235 (63), 236 (35), 237 (5), 238 (42), 239 (10), 246 (1), 247 (2), 263 (3), 266 (1), 300 (3), 301 (1), [2] (2), [4] (4), [5] (8), [6] (5), [7] (12), [9] (1), [56] (2). The notation [56], for example, refers to p. 56 in the second section, of 76 p., in the volume. There are also changes 298(6), 299(2).

In earlier pages of *MTAC* a number of these errors have been already listed. References to these, a discussion of other errors of more than a unit in the final figure, as well as a list of errors in the 1909 and 1933 editions, are given elsewhere in this issue, MTE **61**, p. 391 f.

Thus information here placed at the disposal of owners of all editions will enable them to increase the value of their editions as tools, while purchasers of the volume under review will have a greatly improved edition of an extraordinarily useful work.

R. C. A.

188[C, D].—*Five-Figure Logarithm Tables containing Logarithms of Numbers and Logarithms of Trigonometrical Functions with Arguments in Degrees and Decimals.* Published for the MINISTRY OF SUPPLY by His Majesty's Stationery Office, London, 1944. iv, 74, XXII, 29–120 p. 15 × 24 cm. 7s 6d.

"This collection of 5-figure tables of logarithms and logarithmic trigonometrical functions has been compiled as a war-time measure, primarily to meet the urgent requirements

of the optical industry. Expediency can be the only justification for the juxtaposition of differing page sizes, of differing styles of types and of differing forms of printing." So begins the Preface by H. J. GOUGH, director-general of Scientific Research and Development, Ministry of Supply, writing in September 1944.

The first section of the volume is a reproduction of p. 2–73 of E. CHAPPELL, *Five-Figure Mathematical Tables . . .*, Edinburgh and London, W. & R. Chambers, 1915. Through an arrangement with the publishers the original stereotype plates were used. These reproduced pages give logarithms of numbers 10 000 to 40 000, and 4 000 to 10 000, with differences and proportional parts. See *MTAC*, p. 195. The next two parts are photolithographic reproductions of German works, under license from the Custodian of Enemy Property. The first of these has the title-page: *Die Logarithmen der Sinus und Tangenten für 0° bis 5° und der Cosinus und Cotangenten für 85° bis 90° von Tausendstel zu Tausendstel Grad als Ergänzung zu C Bremiker's 5 stelligen Logarithmentafeln, herausgegeben von* M. von ROHR . . . Berlin, Weidmannsche Buchhandlung, 1900, and the table occupies pages I–XX. The final part of the volume has the title page: *Dr. C. Bremikers Logarithmisch-Trigonometrische Tafeln mit fünf Decimalstellen, besorgt von* . . . A. KALLIUS. *Sechzehnte Stereotyp-Auflage*, Berlin, Weidmannsche Buchhandlung, 1925. The pages 30–119 contain five-figure logarithms of sines, tangents, cotangents and cosines, for the angles 0(0°.01)45°. On the final page are values of constants.

The original edition of these tables of CARL BREMIKER (1804–1877) appeared in 1872. Beginning with the third edition in 1880, A. Kallius was the editor. There was a seventeenth edition in 1937. The third unchanged edition, of the supplement by Louis Otto Moritz von Rohr (1868–), appeared in 1933.

Explanations of any kind in connection with the tables were naturally regarded as unnecessary. Some American libraries will doubtless be glad of the opportunity offered for purchasing this volume so as to get a copy of Rohr's table.

Workers in the optical industry in this country, where computing machines are so freely used, are not likely to find any use for the volume under review. Rather would they turn to such a volume as the seven-place table of natural trigonometric functions for every thousandth of a degree, prepared by Peters in 1918 for the Goerz optical establishment, now readily available in an American edition. See *MTAC*, p. 12–13.

R. C. A.

189[D].—HERMANN BRANDENBURG, *Sechsstellige trigonometrische Tafel alter Kreisteilung für Berechnungen mit der Rechenmaschine, enthaltend die unmittelbaren oder natürlichen Werte der vier Winkellinien-Verhältnisse: Sinus, Tangens, Cotangens und Cosinus des in* 90° *und* 60′ *geteilten Einheits-Viertelkreises in Unterschieden von* 10 *zu* 10 *Sekunden nebst zwei Vortafeln mit Einzelsekundenwerten der Cotangente und erweiterten Zehnsekundenwerte des Sinus und der Tangente kleiner Winkel. . . . Das Vorwort in deutscher, englischer, französischer, spanischer, holländischer, russischer, und japanischer Sprache*, Leipzig, Lorenz, 1932. Back of the title page is, in part, *Copyright vested in the Alien Property Custodian, 1944, pursuant to law. Published and distributed in the Public Interest by authority of the Alien Property Custodian under License No. A.407. Lithoprinted by Edwards Brothers, Inc., Ann Arbor, Michigan, U. S. A.*, 1945. xxii, 304 p. 20.2 × 28 cm. $5.00.

In 1923 Brandenburg published seven-place trigonometric tables containing natural values of the Sine, Tangent, Cotangent, and Cosine at interval 10″, with differences and proportional parts. There were also an introductory table of Cotangent up to 6°, and of Tangent from 84° to 90°, both at interval 1″. The construction of this work was based on the 10-place trigonometric table of RHÆTICUS, 1596, 1607, and the 15-place trigonometric

table of PITISCUS, 1613 (see *MTAC*, p. 8–9, and A. DEMORGAN, *English Cyclopaedia*, ed. C. Knight, *Arts and Sciences*, v. 7, London, 1861, article "Table"). But this edition of Brandenburg's work contained over 400 errors. All of these were eliminated in the second edition of 1931, which is "practically free from error." This work was prepared chiefly for computers working with calculating machines.

From the material of these editions Brandenburg prepared six-place tables by rounding off the last decimal, and thus appeared the original of the work now under review. Vortafel 1 is of Cotangent to 3° (or of Tangent 87° to 90°) at interval 1″, to 7S; and Vortafel 2 is of Sine and Tangent to 1°, at interval 10″, mostly to 7 or 8D. The last five pages contain miscellaneous numerical data and tables. Nine errata discovered in this volume by L. J. C. were published in *MTAC*, p. 162. Six of these, and two other corrections on p. 64, are listed on the back of the dedication page (p. iv). The two corrections not previously listed in *MTAC* are as follows:

Diff. following cot 5° 42′ 30″, *for* 4903, *read* 4898
P.P. for 9/10 of 490, *for* 441.0, *read* 441,0.

The volume is well reproduced, and bound in light-blue buckram, with title in black ink along the back.

Brandenburg well remarks that a "6-figure table has one advantage that must not be underestimated, namely that it suffices for the greater part of the calculations encountered in the solution of triangles, and because of its convenience in manipulation is particularly advantageous when five or fewer figures are sufficient, in which case the unused figures are rejected. Further the owner of a 7-figure table generally equips himself with a 6-figure table also, whereas the converse is rarely the case. The 6-figure table is undoubtedly especially beloved in the computing profession, and can be pronounced the favorite among trigonometrical tables."

R. C. A.

190[F].—L. ALAOGLU & P. ERDÖS, "On highly composite and similar numbers," Amer. Math. So., *Trans.*, v. 56, Nov. 1944, p. 448–469. 19.5 × 27.5 cm.

This paper contains two tables (p. 467–469) having to do with so-called abundant numbers. If the sum $\sigma(n)$ of the divisors of n exceeds $2n$, then n is called *abundant*. In case $\sigma(n)>\sigma(m)$ for all $m<n$, then n is called *highly abundant*. In case $\sigma(n)/n>\sigma(m)/m$ for all $m<n$, then n is called *superabundant*.

The first table lists all highly abundant numbers $\leqslant 10080$. This table errs by including the non-highly abundant number 2280 and by excluding the highly abundant numbers 1920 and 2160. The number of highly abundant numbers $\leqslant 10080$ is thus 84, not 83. The table also contains the minor errata:

$n = 1080$ *for* $2^3.3.5$, *read* $2^3.3^2.5$
$n = 4620$ *for* $\sigma(n) = 16120$, *read* 16128.

This table was based on Glaisher's tables[1] of $\sigma(n)$ for $n \leqslant 10^4$.

The second table lists all superabundant numbers $< 10^{18}$ in their factored form together with values to 3D of $\sigma(n)/n$ in each case. These numbers are similar to Ramanujan's[2] highly composite numbers which have more divisors than any smaller number.

D. H. L.

[1] J. W. L. GLAISHER, Table I in BAASMTC, *Mathematical Tables, volume VIII, Number-Divisor Tables*, Cambridge, 1940.

[2] S. RAMANUJAN: "Highly composite numbers," London Math. So., *Proc.*, s. 2, v. 14, 1915, p. 347–409; *Collected Papers*, Cambridge, 1927, p. 78–128.

191[L].—R. P. BELL, "Eigen-values and eigen-functions for the operator $d^2/dx^2 - |x|$," *Phil. Mag.*, s. 7, v. 35, Sept. 1944, p. 584–585. 16.9 × 25.3 cm.

T. I gives values of $J_{2/3}(z)$ and $J_{-2/3}(z)$, for $z = [0(.02)1; 4D]$. These were calculated from the following series:

$$J_{2/3}(z) = .69783z^{2/3}(1 - .1500z^2 + .00703z^4 - .00016z^6 + \cdots), \text{ and}$$
$$J_{-2/3}(z) = .59255z^{-2/3}(1 - .7500z^2 + .07031z^4 - .00251z^6 + .000047z^8 - \cdots).$$

Interpolation in T. I gave the first zero of $J_{2/3}(z) - J_{-2/3}(z)$ as $z = .6856$, not quite agreeing with the value given by H. M. MACDONALD, R. So. London, *Proc.*, v. 90A, 1914, p. 54, where we find .6854. The eigen-values λ_n of $\left(\frac{d^2}{dx^2} - |x| + \lambda\right)\psi = 0$ are given by

$$J_{2/3}(\tfrac{2}{3}\lambda_n^{3/2}) - J_{-2/3}(\tfrac{2}{3}\lambda_n^{3/2}) = 0, \; n \text{ even},$$
$$J_{1/3}(\tfrac{2}{3}\lambda_n^{3/2}) - J_{-1/3}(\tfrac{2}{3}\lambda_n^{3/2}) = 0, \; n \text{ odd}.$$

The first six values of λ_n are given, $n = [0(1)5; 3D]$, in T. II.

In T. III are the eigen-values λ_n of $\left(\frac{d^2}{dx^2} - |x^q| + \lambda\right)\psi = 0$ for $q = 1, 2, 4, \infty$, $n = [0(1)5; 2 \text{ or } 3D]$.

The error in Watson's *Bessel Functions*, p. 751, noted by J. C. P. MILLER, *MTAC*, p. 367, was already published by Bell, on p. 584.

R. C. A.

192[L].—R. CLARK JONES, "On the theory of the directional patterns of continuous source distributions on a plane surface," Acoustical So. Amer., *J.*, v. 16, Jan. 1945, p. 158–171. 19.8 × 26.6 cm.

Computations are made for distributions of finite extent with a distribution function of type

$$A(u) = (1 - u^{2m})^n$$

with 10 different choices of (m, n) for both the circular and the linear cases. These choices are:

$$(1, -1), (1, 0), (1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (3, 1), (4, 1).$$

There are the following 21 tables: T. I.—$J_0(\alpha)$ and $J_0^2(\alpha)$ to 5S, $10\log J_0^{-2}(\alpha)$ to 4S, for $\alpha = 0(.4)20$. Also first 12 values of α for maxima of $J_0^2(\alpha)$ (zeros of $J_1(\alpha)$) to 3D. T. II.—$Q_N = 2J_1(\alpha)/\alpha$ and Q_N^2 to 6S, $10\log Q_N^{-2}$ mostly 4S, $\alpha = 0(.1)20$. Also first 12 values of α for maxima of Q_N^2 (zeros of $J_2(\alpha)$) to 3D. T. III.—$Q_N = 8J_2(\alpha)/\alpha^2$ and Q_N^2 to 5S, and $10\log Q_N^{-2}$ to 4S, $\alpha = 0(.4)20$. Also first 11 values of α for maxima of Q_N^2 (zeros of $J_2'(\alpha)$) to 3D. T. IV.—$Q_N = 48J_3(\alpha)/\alpha^3$ and Q_N^2 to 5S, and $10\log Q_N^{-2}$ to 2D, $\alpha = 0(.4)20$. Also first 11 values of α for maxima of Q_N^2 (zeros of $J_4(\alpha)$), to 3D. T. V.—$Q_N = 384J_4(\alpha)/\alpha^4$ and Q_N^2 to 5S, and $10\log Q_N^{-2}$ to 2D, $\alpha = 0(.4)20$. Also first 10 values of α for maxima of Q_N^2 (zeros of $J_5(\alpha)$), to 3D. T. VI.—$Q_N = 3840J_5(\alpha)/\alpha^5$ and Q_N^2 to 5S, and $10\log Q_N^{-2}$ to 4S, $\alpha = 0(.4)20$. Also first 10 values of α for maxima of Q_N^2 (zeros of $J_6(\alpha)$). T. VII.—$Q_N = 12J_2(\alpha)/\alpha^2 - 24J_3(\alpha)/\alpha^3$ and Q_N^2 to 5S, and $10\log Q_N^{-2}$ to 4S, $\alpha = 0(.4)20$. T. VIII.—$Q_N = 120J_3(\alpha)/\alpha^3 - 720J_4(\alpha)/\alpha^4 + 1440J_5(\alpha)/\alpha^5$ and Q_N^2 to 5S, and $10\log Q_N^{-2}$ to 4S, $\alpha = 0(.4)20$. T. IX.—$Q_N = 16J_2(\alpha)/\alpha^2 - 64J_3(\alpha)/\alpha^3 + 128J_4(\alpha)/\alpha^4$, and Q_N^2 to 4S, and $10\log Q_N^{-2}$ to 4S, $\alpha = 0(.4)20$. T. X.—$Q_N = 20J_2(\alpha)/\alpha^2 - 120J_3(\alpha)/\alpha^3 + 480J_4(\alpha)/\alpha^4 - 960J_5(\alpha)/\alpha^5$ and Q_N^2 to 4S, and $10\log Q_N^{-2}$ to 4S. T. XI.—$Q_N = \sin\beta/\beta$ to 6S, and Q_N^2 to 5S, and $10\log Q_N^{-2}$ to 5S, $\beta = 0(.4)20$. Also first 5 values of β, to 2D, for maxima of Q_N^2. T. XII.—$Q_N = 3(\pi/2\beta)^{\frac{1}{2}}J_{3/2}(\beta)/\beta$ to 6S, Q_N^2 to 5S, and $10\log Q_N^{-2}$ to 5S, $\beta = 0(.4)20$. Also first 5 values of β, 2–3D, for maxima of Q_N^2. T. XIII.—$Q_N = 15(\pi/2\beta)^{\frac{1}{2}}J_{5/2}(\beta)/\beta^2$ to 6S,

and Q_N^2 to 5S, and $10\log Q_N^{-2}$ to 4S, $\beta = 0(.4)20$. Also first 4 values of β, to 2D, for maxima of Q_N^2. T. XIV.—$Q_N = 105(\pi/2\beta)^{\frac{1}{2}}J_{7/2}(\beta)/\beta^3$ to 6S, Q_N^2 to 5S, $10\log Q_N^{-2}$ to 4S, $\beta = 0(.4)20$. Also first 4 values of β, to 2D, for maxima of Q_N^2. T. XV.—$Q_N = 945(\pi/2\beta)^{\frac{1}{2}}J_{9/2}(\beta)/\beta^4$ to 6S, Q_N^2 to 5S, $10\log Q_N^{-2}$ to 4S, $\beta = 0(.4)20$. T. XVI.—$Q_N = (\pi/2\beta)^{\frac{1}{2}}[5J_{3/2}(\beta)/\beta - 10J_{5/2}(\beta)/\beta^2]$ and Q_N^2 to 5S, $10\log Q_N^{-2}$ to 4S, $\beta = 0(.4)20$. T. XVII.—$Q_N = (\pi/2\beta)^{\frac{1}{2}}[45J_{5/2}(\beta)/\beta^2 - 270J_{7/2}(\beta)/\beta^3 + 540J_{9/2}(\beta)/\beta^4]$, and Q_N^2 to 5S, and $10\log Q_N^{-2}$ to 4S, $\beta = 0(.4)20$. T. XVIII.—$Q_N = (\pi/2\beta)^{\frac{1}{2}}[7J_{3/2}(\beta)/\beta - 28J_{5/2}(\beta)/\beta^2 + 56J_{7/2}(\beta)/\beta^3]$, and Q_N^2 to 5S, $10\log Q_N^{-2}$ to 4S, $\beta = 0(.4)20$. T. XIX.—$Q_N = (\pi/2\beta)^{\frac{1}{2}}[9J_{3/2}(\beta)/\beta - 54J_{5/2}(\beta)/\beta^2 + 216J_{7/2}(\beta)/\beta^3 - 432J_{9/2}(\beta)/\beta^4]$, and Q_N^2 to 5S, and $10\log Q_N^{-2}$ to 4S, $\beta = 0(.4)20$. T. XXI.—$Q_N = \cos\beta/[1 - (2\beta/\pi)^2]$ and Q_N^2 to 5S, $10\log Q_N^{-2}$ to 4S, $\beta = 0(.4)20$. T. XXII.—$Q_N = \sin\beta/[\beta(1 - (\beta/\pi)^2)]$ and Q_N^2 to 5S, $10\log Q_N^{-2}$ to 4S, $\beta = 0(.4)20$.

R. C. A. & H. B.

193[L].—National Defense Research Committee, Division 6, *Scattering and Radiation from Circular Cylinders and Spheres*, prepared by A. N. Lowan of the N. Y. M. T. P., & by P. M. Morse, H. Feshbach, M. Lax, of the M. I. T. Underwater Sound Laboratory. Report February 1945. iii + 124 leaves, 21.5 × 28 cm. Printed on only one side of each leaf, by the photo-offset process from manuscript. These tables are available only to certain Government agencies and activities.

In problems of wave motion external to spheres and cylinders the Bessel functions of the first and second kinds are of importance. The tables 1–5, 8–12 in this report are all for $x = 0(.1)10$ with 2D, but the number of significant figures generally varies from 3 to 5. In tables 6, 7, 13 and 14, 6S are given and the ranges are $x = 0(.1)10$, $n = 0(1)20$. The same range for n is used for all the other tables except 10 and 11 where $n = 0(1)15$. The quantities tabulated are:

Cylindrical Functions	*Spherical Functions*
1. $\alpha_n(x) = \tan^{-1}[-xJ_n'(x)/J_n(x)]$	8. $\alpha_n(x) = \tan^{-1}[-xj_n'(x)/j_n(x)]$
2. $\beta_n(x) = \tan^{-1}[-xN_n'(x)/N_n(x)]$	9. $\beta_n(x) = \tan^{-1}[-xn_n'(x)/n_n(x)]$
3. $\delta_n(x) = \tan^{-1}[-J_n(x)/N_n(x)]$	10. $\delta_n(x) = \tan^{-1}[-j_n(x)/n_n(x)]$
4. $\delta_n'(x) = \tan^{-1}[-J_n'(x)/N_n'(x)]$	11. $\delta_n'(x) = \tan^{-1}[-j_n'(x)/n_n'(x)]$
5. $\gamma_n(x) = \tan^{-1}[\tan\delta_n \sec\alpha_n \cos\beta_n]$	12. $\gamma_n(x) = \tan^{-1}[\tan\delta_n \sec\alpha_n \cos\beta_n]$
6. $C_n(x) = \lvert J_n(x) + iN_n(x)\rvert$	13. $D_n(x) = \lvert j_n(x) + i\,n_n(x)\rvert$
7. $C_n'(x) = \lvert J_n'(x) + iN_n'(x)\rvert$	14. $D_n'(x) = \lvert j_n'(x) + i\,n_n'(x)\rvert$

The function $N_n(x)$ is the Bessel function of the second kind denoted by $Y_n(x)$ in *MTAC*. The functions $j(z)$, $n(z)$ are defined by the equations

$$j_n(z) = (\pi/2z)^{\frac{1}{2}}J_{n+\frac{1}{2}}(z), \qquad n_n(z) = (\pi/2z)^{\frac{1}{2}}N_{n+\frac{1}{2}}(z).$$

A useful list of properties of the functions is given. In the table of contents the expressions for the spherical functions $\delta_n(x)$ and $\delta_n'(x)$ in 10–11 are given incorrectly.

H. B.

194[L].—Îa. M. Serebriĭskiĭ, "Obtekanie tel vrashcheniîa" [Flow past bodies of revolution], Akad. Nauk, SSSR, Otdelenie tekhnichnikh Nauk, Institut Mekhaniki, *Prikladnaîa Matematika i Mekhanika*, *Applied Mathematics and Mechanics*, v. 8, June 1944, p. 103, 104. 16.7 × 25.5 cm.

There is a table of Legendre functions $P_n(\mu)$, $dP_n(\mu)/d\mu$, for $n = 1(1)5$, $\mu = [-1(.05) -.9(.1)+.9(.05)1$; 4D], where $\mu = \cos\theta$; θ is given to the nearest 1′ for each value of μ.

There are also graphs of $Q_n(\lambda) = Q_n(\cosh\psi)$, and of $\delta_n(\psi)$, for $\psi = .1(.05).35$, $n = 1(1)5$, where $dQ_n(\lambda)/d\lambda = \delta_n(\psi) - 1/\psi^2$.

H. B.

MATHEMATICAL TABLES—ERRATA

Reference has been made to Errata under RMT**187** (Jahnke & Emde), **189** (Brandenburg), **190** (Alaoglu & Erdös), **191** (Watson), **193** (N. D. R. C.); N**30** (Girard), **36** (Jacobi).

61. E. JAHNKE & F. EMDE, *Tables of Functions.* Editions as in RMT **113**, p. 106f. The errors here listed are almost wholly supplementary to those discussed p. 108–109, 136–137, 161, 196, 198, 204 (corrigenda, p. 109, 161), 286, 287, 294, 308 (corrigenda, p. 137), 325.

The popularity on both sides of the Atlantic of the Dover Publications reprint (1943) of these well-known tables makes a list of their errors timely. The following lists do not pretend to be complete, but as they put on record the results of systematic examination of some of the tables, they may well relieve anxieties about the correctness of certain tables. I wish to place on record the great share that J. C. P. MILLER has had in these examinations, and in perfecting this list. While references have been previously made in *MTAC* to contributions of NYMTP we present here a complete list of other discovered errors exceeding a unit in the last decimal place.

The original tables (1909) were followed by a flood of corrections, mostly in *Arch. Math. u. Physik.* That perfection was not attained in the later editions is evident. The surviving compiler (Emde) is not entirely to blame, as many of the tables have been taken from diverse sources (always freely acknowledged); it was obviously impossible to recompute every such table.

Table I isolates errors that occur in the 1909 edition only. Some more known, but obvious or unimportant, errors have been omitted. In both lists, with few exceptions (the principal being in the table of powers in the 1933 edition), errors of a unit in the last decimal are not given, although often statistics of their frequency will be found in the notes. The practical view-point is that the labour of making such end-figure alterations would not be justified; nor is it fair to classify them as errors if the author has not claimed absolute accuracy in the last decimal.

One of my most treasured possessions is the late J. R. Airey's personal copy of the 1909 edition of these tables. It is covered with beautifully written notes, references and corrections, and has 120 pages of related formulae, etc. pasted in—all neatly transcribed, with their sources if they are taken from published works. Whenever Airey visited me, he *always* came hugging this book mysteriously! It is now shared with J. C. P. Miller, on whom Airey's mantle has fallen.

In the date column, . . . denotes that the quantity to be corrected is not given; *C* denotes that it is correct. Page numbers in square brackets in the 1938 column are a reminder that these pages are at the end of the reprint (1943) by Dover Publications.

In the column "authority," *A* = Airey (marked in his personal copy, with no source ascribed), *C* = Comrie, *M* = Miller, *A.M.*, followed by a number, denotes *Archiv für Mathematik und Physik*, dritte Reihe, and the volume number. The complete references here are:

v.	year	pages	authors
15	1909	372–4	Bruns, Egloff, Emde, Greenhill, Kritaniger, Witt.
16	1910	285–6	Baruch, Debye, Emde, Goldscheider, Naetsch, Russell.
17	1910	286	Dinnik, Goldscheider, Jahnke.
18	1911	204	Dinnik.
18	1911	374	Emde, Jahnke, Schaefer.
20	1912	95	Escott.
21	1913	191	Emde, Escott, Jahnke, Nagaoka.
22	1913	95	Wagner.
22	1914	285	Airey.
22	1914	376	?
24	1915	94	Kalähne.

TABLE I—Errors in 1909 edition only

1933–8	1909			*for*	*read*	authority
C	1	$x \tan x$	$x = 0.2$	.04045	.04054	M
			$x = 1.4$	8.2627	8.1170	
			$x = 1.5$	21.1523	21.1521	
		$\frac{\tan x}{x}$	$x = 0.2$	1.0112	1.0136	M
			$x = 0.5$	1.0927	1.0926	
			$x = 1.2$	2.1434	2.1435	
			$x = 1.4$	4.2157	4.1413	
C	3	$\cos x \cosh x = 1$. $x = 0$ is also a root				*A.M.*, 18
...	6	e^x for $x = 6.0$		403.00	403.43	*A.M.*, 20
...	10	Penultimate line		$-\mathbf{Ar}\,\mathbf{Sin}\,ix$	$-i\,\mathbf{Ar}\,\mathbf{Sin}\,ix$	*A.M.*, 16
...	11	Line−10		$\sin 2x$	$\mathbf{Sin}\,2x$	*A.M.*, 16
C	12	Last line, middle integral		$\mathbf{Tg}\frac{x}{2}$	$\mathbf{Tg}\,x$	*A.M.*, 22
C	12	$d\,\mathbf{Ar}\frac{1}{\mathbf{Tg}\,x}$		$\frac{1}{\mathbf{Tg}\,x}$	$\mathbf{Ctg}$	*A.M.*, 22
...	19	Last line of Section 1		$Ei(ix)\,Ci(x)$	$Ei(ix) = Ci(x)$	*A.M.*, 16
...	20	Line 4		$-\frac{1}{2}\pi\cos 2x$	$\frac{1}{2}\pi(1-\cos 2x)$	*A.M.*, 16
C	22, 23	$Ci(x)$ for $x = 40(5)75$, 120(10)170, insert an additional cipher after the decimal point				M
C	26	$S(1.6)$		.6383	.6389	(12)
		$C(1.8)$		.3363	.3337	
		$S(2.3)$		.5525	.5531	
...	27	Line 2		$\frac{1}{2\sqrt{2}}$	$\frac{\pi}{2\sqrt{2}}$	*A.M.*, 16
		Line 7		2^{∞}	$2^{\frac{1}{2}\pi}$	*A.M.*, 16
		Line 3 of Section 5		See note		A(1)
...	28	Line 8, denominator		$x - \frac{1}{2}$	$x + \frac{1}{2}$	*A.M.*, 16
		Line 9, denominator		$\pi^3(\frac{1}{2})$	$\pi(\frac{1}{2})$	*A.M.*, 16
...	29	Line 4, replace last term by			$+\frac{1}{12}x$	*A.M.*, 16
C	29	Legend of Fig. 8		Gamma-funktion	Fakultät	A
...	36	Diagram		$\frac{1}{3}\Phi^{\mathrm{VI}}$	$\frac{1}{3}\Phi^{\mathrm{IV}}$	*A.M.*, 15
		Diagram		$\frac{1}{3^2}\Phi^{\mathrm{IV}}$	$\frac{1}{3^2}\Phi^{\mathrm{VI}}$	
...	44	Characteristic of logs in $\phi = 0°, 5°, 10°$		1	$\bar{1}$	*A.M.*, 15
	45	$r = 48$ $\phi = 10°$		0	$\bar{1}$	M
...	47	Line 2, last term in numerator		$\mathrm{d}\,v$	$\mathrm{dn}\,v$	*A.M.*, 16
...	51	Line−3		$\mathbf{Amp}\,x$	$\mathbf{Amp}\,u$	*A.M.*, 16
...	67	$\log q$ for $\alpha = 64°\ 5'$		.4087	.0087	*A.M.*, 21
		$\log q$ for $\alpha = 69°\ 55'$		.1195	.1159	
C	71	Upright side. Denominators		dv	πdv	*A.M.*, 16
				dv_1	πdv_1	
...	72	Legend of Fig. 22		1.53995	1.52995	A
...	78	Beginning of line 7		$\frac{dF}{dk}$	$\frac{dK}{dk}$	*A.M.*, 22
...	78	Line 1 of Section 3; Niven's initials		M	W.D.	*A.M.*, 16

TABLE I—(*Continued*)

1933–8	1909		*for*	*read*	authority
C	83	$P_1(.03)$	.3000	.0300	*A.M.*, 15
C	85	$P_7(1°)$	.9955	.9957	C(2)
		$P_7(3°)$	.9617	.9620	
		$P_6(7°)$	.8476	.8492	
		$P_7(7°)$	.7986	.8016	
		$P_7(13°)$	.3940	.3980	
		$P_7(18°)$	.0289	.0248	
		$P_3(19°)$	.1347	.1353	
		$P_7(19°)$	.0443	.0433	
C	86	$P_7(21°)$	.1662	.1664	C(2)
		$P_3(25°)$	.2053	.2040	
		$P_7(25°)$	.3463	.3441	
		$P_3(40°)$	.3234	.3236	
		$P_7(40°)$	.1003	.1006	
		$P_4(47°)$	.4252	.4227	
		$P_6(47°)$	.0645	.0665	
		$P_7(47°)$	.2054	.2028	
		$P_7(57°)$	.2949	.2947	
C	87	$P_6(71°)$	.1786	.1791	C(2)
		$P_7(71°)$	.1811	.1808	
		$P_6(73°)$	.1144	.1136	
		$P_7(73°)$	.2347	.2352	
		$P_3(74°)$	.0795	.0788	
		$P_7(74°)$	.2559	.2563	
		$P_2(76°)$	.4112	.4122	
		$P_6(76°)$	.0076	.0070	
		$P_7(76°)$	.2848	.2850	
		$P_3(77°)$	.0284	.0290	
		$P_7(77°)$	.2919	.2921	
		$P_7(88°)$	.0735	.0755	
...	94	Second formula for $Y_1(x)$	$J_1(x)$	$J_1(x)(\ln x - 1)$	*A.M.*, 16
...	96	$\lim_{x=0} N_1(\pm ix)$	See note (3)		*A.M.*, 16
C	99	Expression for large roots of J_0	.50661	.050661	*A.M.*, 16
		Expression for large roots of J_1	.15399	.015399	
...	100	Line − 3	185	191	A
...	102	Coefficient of $\cot g^2\tau$ in $A_2(\tau)$	$\frac{7}{576}$	$\frac{77}{576}$	*A.M.*, 17
...	103	Coefficient of $\cot g^2\tau$ in $B_2(\tau)$			
		Series for $J_p(x)$	$\frac{1}{\pi}$	$\frac{1}{2\pi}$	*A.M.*, 17
...	106	Smallest root of $J_{-1/3}(x)=0$	1.88	1.8663:	A(4)
		$D(10)$	.078	.068	A
C	111	$J_0(0.62)$	.9052	.9062	*A.M.*, 22
C	115	$J_0(5.90)$	.1120	.1220	*A.M.*, 22
C	118	$-J_1(10.30)$	−	+	*A.M.*, 16
...	125	$Y_0'(5.2)$	.0816	.0846	A(5)
		$Y_0'(8.1)$	.1806	.1804	
		$Y_0'(8.5)$	.0091	.0094	
		$Y_0'(8.6)$	.0336	.0333	

TABLE I—(*Continued*)

1933–8	1909		for	read	authority
		$Y_0'(8.7)$	.0751	.0753	
		$Y_0'(8.8)$	.1157	.1160	
		$Y_0'(8.9)$	.1548	.1551	
		$Y_0'(9.0)$	.1921	.1923	
		$Y_0(9.5)$	.2463	.2465	
		$Y_0'(9.5)$	.3381	.3378	
		$Y_0'(9.6)$	.3574	.3571	
		$Y_0(9.7)$	.1755	.1752	
		$Y_0'(9.7)$	.3729	.3727	
		$Y_0'(9.8)$	.3846	.3844	
		$Y_0(9.9)$	.0982	.0984	
		$Y_0'(10.2)$	.3931	.3922	
...	127	$K_1(0.20)$	5.2209	5.2211	A(6)
		$K_1(0.65)$	1.8483	1.8485	
...	128	$K_1(2.4)$	.1580	.1578	A(6)
		$K_1(8.1)$	.2093	.2091	
		$K_1(8.5)$	.0408	.0411	
		$K_1(8.6)$	.0020	.0017	
		$K_1(8.7)$	.0438	.0440	
		$K_1(8.8)$	.0851	.0854	
		$K_1(8.9)$	.1251	.1255	
		$K_1(9.0)$	.1637	.1639	
		$K_1(9.5)$	.3194	.3192	
		$K_1(9.6)$	.3412	.3410	
		$K_0(9.7)$	.2012	.2009	
		$K_1(9.7)$	.3594	.3592	
		$K_0(9.9)$	.1261	.1263	
		$K_1(10.2)$	.3923	.3930	
C	129	$-N_1(0.1)$	7.0317	6.4590	M(7)
		$-N_1(0.2)$	3.3235	3.3238	M
		$N_0(3.9)$	.0237	.0234	M
		$-N_1(5.2)$	.0773	.0792	A
C	130	$-N_1(8.8)$	.0541	.0544	M(7)
		$N_0(9.9)$	.0800	.0804	M
		$-N_1(10.2)$	.2507	.2502	A
C	137	$x = 5.1$ to 6.0. Imaginary part	−	+	*A.M.*, 17
...	138	$x = 3.1$	1.8918	0.8919	
		$x = 3.3$	0.3556	1.3556	*A.M.*, 18
C	139	$x = 6.0$. Real part of $H_0^{(1)}$	+	−	*A.M.*, 16
...	140	$x = 2.4$. Real part	−	+	*A.M.*, 18
...	146	Table of ber x, etc.	See note (8)		*A.M.*
C	153	$J_7(13)$	+	−	A
...	164	Line −4	XVI	XXVI	A
...	166	Right-hand side of fifth integral	See note (9)		*A.M.*, 16
...	167	Last differential equation	$\frac{p^2}{x^2}$	$\frac{p}{x^2}$	*A.M.*, 16
...	168	c) Second equation	$-\frac{b^2}{16p^2}$	$-\frac{b^2}{16p^4}$	*A.M.*, 17
...	175	Reference to W.D. Niven	77	78	*A.M.*, 20
...	175	Reference to P. A. Hansen	170	169	*A.M.*, 15

TABLE II

1938	1933	1909		for	read	authority
9	86	22	$Si(65)$	1.5775	1.5792	M(10)
10	87	...	C_3	0,067 353 011	0,067 352 301	A. Fletcher (11)
			$H(x)$	$-\frac{571}{248\,832\,x^4}$	$-\frac{571}{2\,488\,320\,x^4}$	A. Fletcher
34	106	...	$\psi_n(x)$	$\Phi_1\left(\frac{x}{2}\right)$	$\Phi_1\left(\frac{x}{\sqrt{2}}\right)$	M
34	109	26	$C(0.1)$	.0999	.1000	(12)
			$C(0.8)$	.7230	.7228	
			$C(1.1)$	.7648	.7638	
			$C(4.5)$	.5258	.5260	
			$C(5.1)$	.4987	.4998	
			$S(7.3)$	.5199	.5189	
35	108	25	$S(1.5)$	.4153	.4155	M
39	113	...	Heading x	.01	.1	M
70	142	62	$E(15°, 54°)$	.9345	.9346	M(13)
74	146	...	Coefficient of e^{17}	1701	1707	A(14)
94	160	...	In first bordered table	$\frac{-d}{ks}$	$\frac{-d}{kc}$	Todd (letter)
103	169	74	σu for $r = 123$	1.0441	1.0438	M
105	171	50	Last integral formula for Weierstrassian functions. After this integral add: where v is defined by $\varphi(v) = \delta/\gamma$	$\frac{\alpha u}{\gamma} -$	$\frac{\alpha u}{\gamma} +$	Todd (letter)
108	174	...	Diagram is in error			(15)
117	183	82	Expression for $dP_7/d\theta$	$\sin 3\theta$	$\sin^3\theta$	*A.M.*, 16 (16)
124	190	88	$dP_n/d\theta$, $n = 3$, $\theta = 31°$	2.0654	2.0656	NYMTP (17)
			$n = 4$, $\theta = 5°$	0.8570	0.8567	
			$n = 4$, $\theta = 64°$	1.6306	1.6300	
			$n = 5$, $\theta = 54°$	2.0173	2.0178	
			$n = 5$, $\theta = 64°$	1.5420	1.5418	
			$n = 5$, $\theta = 67°$	1.1187	1.1183	
			$n = 5$, $\theta = 83°$	1.4831	1.4827	
			$n = 6$, $\theta = 23°$	3.0917	3.0902	
			$n = 6$, $\theta = 48°$	2.3715	2.3713	
			$n = 6$, $\theta = 57°$	1.1936	1.1934	
			$n = 6$, $\theta = 83°$	1.4487	1.4484	
			$n = 7$, $\theta = 70°$	1.984	1.934	A
C	223	...	$J_{1/4}(4.8)$	.3250	.3255	M(18)
			$J_{3/4}(4.8)$	.3556	.3518	
			$J_{-1/4}(4.8)$	.1149	.1158	
			$J_{-3/4}(4.8)$	.1803	.1791	
			$J_{-1/4}(5.4)$	.0942	.0940	
			$J_{-3/4}(5.4)$	.3085	.3082	
164	223	...	$J_{1/4}(7.4)$	.2925	.2923	NYMTP(19)
			$J_{1/4}(7.6)$	.2873	.2869	
			$J_{1/4}(7.8)$	.2711	.2704	
			$J_{1/4}(8.0)$	.2449	.2436	
			$J_{3/4}(6.8)$	.0447	.0449	
			$J_{3/4}(7.8)$	.2602	.2604	
			$J_{3/4}(8.0)$	.2749	.2752	

TABLE II—(*Continued*)

1938	1933	1909		*for*	*read*	authority
			$J_{-1/4}(.2)$	1.4310	1.4319	
			$J_{-3/4}(3.2)$	.3943	.3945	
168	239	147	$p = 2, n = 9$	30.571	30.569	A, M(20)
			$p = 3, n = 3$	13.017	13.015	
			$p = 3, n = 9$	32.050	32.065	
			$p = 4, n = 1$	7.586	7.588	
			$p = 4, n = 9$	33.512	33.537	
			$p = 5, n = 1$	8.780	8.771	
			$p = 5, n = 4$	18.982	18.980	
			$p = 5, n = 5$	22.220	22.218	
			$p = 5, n = 9$	34.983	34.989	
183	253	...	$x = 6.4$	$d = 193$	$d = 103$	M
206	276	163	$p = 0$ and 1. Delete all bracketed digits			M
212	...	...	$\Omega_0(0.20)$	.1266	.1268	M(21)
216	...	...	$\Omega_0(10.35)$	394	294	M
			$\Omega_0(13.25)$	135	235	
218	...	...	$S_0(4.41)$	273	278	M(21)
220	...	...	$S_0(8.17)$	136	236	M
228	280	...	$I_0(6.4) = J_0(6.4i)$	96.98	96.96	NYMTP
235	285	...	$I_{1/3}(1.4) = i^{-1/3}J_{1/3}(1.4i)$	1.4002	1.4000	NYMTP(22)
			$I_{1/3}(3.6)$	7.8727	7.8729	
			$I_{1/3}(3.8)$	9.3473	9.3470	
			$I_{1/3}(4.0)$	11.1136	11.1138	
			$I_{-1/3}(2.2)$	2.5226	2.5627	
			$I_{-1/3}(3.0)$	4.9710	4.7754	
			$I_{-1/3}(3.6)$	7.8831	7.8827	
			$I_{-1/3}(3.8)$	9.3546	9.3548	
			$I_{-1/3}(4.0)$	11.1204	11.1201	
			$I_{-1/3}(6.0)$	65.55	66.55	
			$I_{2/3}(2.4)$	2.6650	2.6648	
			$I_{2/3}(2.8)$	3.7343	3.7345	
			$I_{2/3}(3.6)$	7.4423	7.4429	
			$I_{2/3}(3.8)$	8.8698	8.8692	
			$I_{2/3}(4.0)$	10.5796	10.5799	
			$I_{2/3}(4.4)$	15.107	15.098	
			$I_{2/3}(5.8)$	53.77	53.71	
			$I_{-2/3}(2.4)$	2.7065	2.7067	
			$I_{-2/3}(2.8)$	3.7002	3.7603	
			$I_{-2/3}(3.4)$	6.2655	6.2657	
			$I_{-2/3}(3.6)$	7.4535	7.4531	
			$I_{-2/3}(3.8)$	8.8841	8.8773	
			$I_{-2/3}(4.0)$	10.5867	10.5864	
			$I_{-2/3}(4.4)$	15.112	15.103	
			$I_{-2/3}(4.8)$	21.621	21.623	
			$I_{-2/3}(8.0)$	418.01	415.01	*A.M.*, 22
263	313	...	$b_0(5.3)$	7.340	7.475	M(23)
			$b_0(9.5)$	108.02	108.00	
			$b_0(9.6)$	115.31	115.30	
266	316	...	$\frac{1}{2}rb_0/b_1$ for $r = 0.5$	006	008	M(24)
300	77	...	(i) Comrie's initials	L.T.	L.J.	C
300	...	...	(i), line 3	.0350	.0400	C

TABLE II—(*Continued*)

1938	1933	1909			*for*	*read*	authority
			(I), line 2		Mittlin	Mifflin	C
[2]	2	...	x^4 for $x = 0.83$		4747	4746	M(25)
			x^6 for $x = 0.88$		4640	4644	
[4]	4	...	x^5 for $x = 1.18$		2289	2288	
			Powers of $x = 1.68$	x^6	2222	2248	
				x^7	3732	3777	
				x^8	6270	6346	
[5]	5	...		x^9	10534	10661	
				x^{10}	17697	17910	
				x^{11}	2973	3009	
				x^{12}	4995	5055	
				x^{13}	8391	8492	
				x^{14}	14097	14267	
				x^{15}	2368	2397	
			x^{10} for $x = 1.40$		2892	2893	
[6]	6	...	x^5 for $x = 2.05$		3620	3621	M(25)
			x^7 for $x = 2.35$		3960	3958	
			Powers of $x = 2.95$	x^6	6590	6591	
				x^7	19441	19443	
				x^8	5735	5736	
[7]	7	...		x^9	16919	16920	
				x^{11}	14723	14725	
				x^{12}	4343	4344	
				x^{13}	12813	12814	
			Powers of $x = 2.85$	x^{10}	3523	3535	
				x^{11}	10041	10076	
				x^{12}	2862	2872	
				x^{13}	8156	8184	
				x^{14}	2324	2333	
				x^{15}	6624	6648	
			x^{11} for $x = 3.10$		2540	2541	
			x^{14} for $x = 3.30$		18164	18163	
[9]	9	...	$x^{0.60}$ for $x = 0.1$		0.251	0.2512	M
[15]	15	...	See note (26)				
[56]	56	13	$\int \frac{dx}{x^2-1}$		$\frac{1}{\mathrm{Tg}\, x}$	Ctg x	*A.M.*, 22

NOTES

(1) Airey has corrected his copy from

$$\sum_{r=2}^{\infty} (1 - s_{2r+1})x^{2r} \text{ to } \sum_{r=1}^{\infty} (1 - s_{2r+1})x^{2r} + (1 - \gamma)$$

(2) In this table many end-figure changes have been made in the 1933 and 1938 editions, expecially in P_6 and P_7; only those greater than one unit are listed here. It will be noticed that, in every case where P_6 is in error, P_7 is in error by $\frac{13}{7} P_1$ times the error in P_6, thus showing that P_7 was found by recurrence, and was not checked by differencing. For further comments on the original table, see *MTAC*, p. 137.

(3) This should read, according to Goldscheider,

$$\lim_{x=0} N_1(\pm ix) = \pm i \frac{2}{\pi x} \pm \frac{ix}{\pi} \ln x - \frac{x}{2} \mp \left(\frac{1}{2} + \ln \frac{2}{\gamma}\right) \frac{ix}{\pi}.$$

(4) See *MTAC*, p. 196 where the value of the root to 15D is also given.

(5) There are also 44 alterations of a unit in the last decimal, taken from Airey's table in B.A.A.S., *Report*, 1914. See also Note (6).

(6) Marked in Airey's personal copy, together with over 120 alterations of a unit in the last figure. See also *A.M.*, v. 18, p. 374 where Jahnke notes these end-figure corrections, based on the B.A.A.S., *Report*, 1911 (computed by Airey). Note the obvious connection between the erroneous arguments here and in the values covered by Note (5), which is explained by

$$Y_n \text{ (J.\&E.)} = \frac{\pi}{2} Y_n \text{ (Watson)} + \ln \frac{2}{\gamma} J_n$$

$$K_n \text{ (J.\&E.)} = -\frac{\pi}{2} Y_n \text{ (Watson)} = G_n \text{ (Airey)}$$

Watson's Y_n is called N_n in Jahnke & Emde.

(7) See the argument 10.2 covered by Notes (5) and (6). Miller has compared this table with Watson's Y_0 and Y_1. Besides the corrections given, he finds 67 errors of a unit, and 6 of two units. Kalähne also points out the error in $-N_1(0.1)$ in *A.M.*, v. 24, p. 94.

(8) See *A.M.*, v. 16 and 21. There are many errors, but it is not worth correcting this table, since correct values may be obtained from the later editions (up to $x = 10$) or from Dwight's *Mathematical Tables* (up to $x = 20$). For a thorough discussion of the errors see *MTAC*, p. 297f.

(9) A further term is needed, namely

$$-\frac{Z_p(\alpha x)\bar{Z}_q(\alpha x)}{p+q}$$

(10) Miller has compared these integrals with the NYMTP tables up to $x = 100$. There are also 5 end-figure errors in the 1933 and 1938 versions of this table.

(11) This error occurs also in Schlömilch, *Comp. d. höh. Analysis*, v. 2, 1879, p. 258, and in Láska, *Sammlung*, 1888–94, p. 272. A. FLETCHER.

(12) There have been "errors within errors" here. A first list was given by Lash Miller and Gordon in *J. Phys. Chem.*, v. 35, 1931, p. 2872. A reference to this paper, and extracts, were given by J. H. Awbery in Phys. So. London, *Proc.*, v. 51, 1939, p. 199. Lash Miller and Gordon failed to note the error at $S(2.3)$ and gave C instead of S for the corrections at 1.6 and 7.3. The first three errors were found independently by Rankin (letter from L. Rosenhead, Feb. 1943). In a 4-figure table prepared by Comrie, $C(1.8)$ appears as .3336, and $S(2.3)$ as .5532. This table shows that there are no other errors exceeding one unit up to $u = 3.3$. This note has reference to T. I, p. 26 and T. II, p. 34, 109, 26.

(13) Traced by means of errors in Legendre's table as given by Heuman (1941).

(14) Also reported by John Todd by letter. Airey has inserted coefficients up to ϵ^{33} by hand in his copy; this one was first written (p. 47, 1909) as 1701 and later changed to 1707.

(15) G. I. Taylor has pointed out that this diagram is completely wrong near $x = -1$. Also it should be symmetrical about $\nu = -\frac{1}{2}$, not $\nu = 0$, as seems suggested.

(16) It is surprising that this published correction (which I have verified) was not incorporated in later editions.

(17) The NYMTP found also 41 errors of a unit in the last decimal. All errors of this table are corrected in the sixth printing of Jahnke & Emde.

(18) Miller has pointed out that the table in 1933 is taken from Dinnik, *A.M.*, v. 21, 1913, p. 324–6, whereas that in 1938 has been corrected to agree with Karas, *Z. angew. Math. u. Mech.*, v. 16, 1936, p. 248, who removed some of the major errors in Dinnik and subtabulated from interval 0.2 to interval 0.1. There are also 5 cases of a change of a unit in the last decimal. See *MTAC*, p. 365–366.

(19) The NYMTP found also 22 errors of a unit in the last decimal. All errors of this table are corrected in the sixth printing of Jahnke & Emde.

(20) Also 18 errors of a unit in the last decimal. Compare *MTAC*, p. 160, 282.

(21) Miller reports that there are also 40 errors of one unit in the last decimal in the table, p. 212, 214, 216. This table is theoretically identical with that on p. 218, 220 and 222, in which only 8 such end-figure errors have been found, none of which corresponds to any of the 40 just mentioned.

(22) The NYMTP found also 37 errors of a unit in the last decimal. All errors of this table are corrected in the sixth printing of Jahnke & Emde.
(23) Miller has also found 9 errors of a unit in the column b_0 and 4 in b_1. See also Note (24).
(24) Miller has also found 17 errors of a unit in the last decimal in the column $\frac{1}{2}rb_0/b_1$ up to $r = 5$. See also Note (23).
(25) This table of powers (for powers greater than 1) has been completely checked by Miller. All errors are given here. The errors in the powers of 1.68 are due to taking $x^6 = 1.66x^5$, and those in 2.85 to taking $x^{10} = 2.84x^9$. They could have been detected by differencing.
(26) Note that d is a mean difference, so that the values of the tabulated functions at 1.00 cannot be accurately obtained; for the upper line of each table this value is 1.0987, and for the lower line 0.4551.

L. J. C.

UNPUBLISHED MATHEMATICAL TABLES

Unpublished mathematical tables are listed in RMT **186** (Lehmer); QR 14.

32[A].—William Pitt Durfee (1855–1941), *Factor Table of the Sixteenth Million.* Unique ms. calculated during the years 1923–1929, and the property of the American Mathematical Society in New York City, since December 1935.

The following description of the ms. was published in *Scripta Math.*, v. 4, 1936, p. 101:

"The table comprises 500 separate sheets, $8\frac{1}{2} \times 14$ inches, each accounting for 2000 numbers, but as the multiples of 2, 3, 5, and 7 are omitted, the actual entries on each sheet number about 416. The entries are in long-hand, in black ink, except that those numbers whose lowest prime factor is 11 have been interpolated in red. They are arranged in parallel columns, three centuries to a column, the last four digits only of each number being written; and opposite each its lowest prime factor. If the number is prime a bar is drawn across the corresponding space in the column of prime factors. The arrangement is thus closely similar to that in the published tables covering the first nine millions (Burckhardt, Dase and Rosenberg, Glaisher).

"At the foot of each column the number of primes for each century is noted, and the total number of primes on preceding sheets, the number on that sheet, and the total. In the lower right-hand margin there are listed the number of entries on preceding sheets, on that sheet, and the total, with a similar notation in red ink for the entries whose least prime factor is 11."

The computations, which were made by the stencil method, have not been generally checked except by the author of the table. However, D. H. L. discovered one error where 15485303 is entered as a prime, when 109 is a factor. The total number of primes in this million is 60,465.

33[A].—U. S. Dept. of Commerce, Weather Bureau, *Table of* $(\log e)/x$, computed by, and in possession of, the Bureau.

The Table of $(\log e)/x$ is for $x = [100(1)999; 6S]$.

L. P. Harrison

Weather Bureau

MECHANICAL AIDS TO COMPUTATION

14[Z].—G. W. King, "Punched-card tables of the exponential functions." *Rev. Sci. Instruments*, v. 15, 1944, p. 349–350.

G. B. Thomas & G. W. King, "Preparation of Punched-card tables of logarithms," *ibid.*, p. 350.

With the rapid increase in recent years of the use of punched-card machines in computations of all kinds, the versatility of the machines has been adapted to many uses.

One obvious method of increasing the number of possible applications has been to prepare cards punched with the values of numerical functions which can be combined with other values by multiplication or addition. Of this type are exponential and logarithmic functions.

The first note indicates the usefulness of punched-card tables of the exponential function. From the relationship, $\exp(a+b) = (\exp a)(\exp b)$, one sees that this function is readily adapted to calculation by punched-card machines, any desired degree of accuracy being easily attained. Thus, the master set of cards might contain values of $\exp \pm n$ over the range $n = 0.00(0.01)12.00$. A second set of 100 cards for $n = 0.0000(0.0001)(0.0099)$ would then multiply the original range a hundredfold.

One of the most important problems in the use of punched cards is the verification of the numbers punched on them. For this purpose the author suggests the use of the formula

$$\sum_{k=0}^{N-\Delta k} e^{-k} \frac{1-e^{-N}}{1-e^{-\Delta k}}.$$

Partial sums of the cards are compared with the values of this function computed for suitably chosen values of Δk.

The second note suggests the use of punched-card tables of $\log x$. The authors suggest that the verification of the cards be made by comparing partial sums of the cards with computed values of $\log n!$ from the formula:

$$\sum_{k=1}^{n} \log k = \log n!.$$

Tables such as those of J. BOCCARDI: *Tables logarithmiques des factorielles jusqu'à 10,0001*, Cavaillon, 1932, or F. J. DUARTE: *Nouvelles tables logarithmiques à 36 decimales*, Paris, 1933, are available for this purpose.

H. T. D.

NOTES

29. EARLY DECIMAL DIVISION OF THE SEXAGESIMAL DEGREE.—In *MTAC*, p. 33, 100 (corrigenda), 129–130, it has been already noted that decimal division of the degree was used by RUFFI in a Latin codex of about 1450, as well as by BRIGGS, in tables prepared before 1633, the idea having been suggested to him by Viète in a work published in 1600. The decimal division of the degree was also advocated by SIMON STEVIN (1548–1620) of Bruges in 1585, in his Flemish work, 1. *De Thiende*, published at Leiden. A facsimile of this 36-page booklet is given in H. BOSMANS, *La 'Thiende' de Simon Stevin*, Antwerp and The Hague, 1924; a French edition of *De Thiende, La Disme*, occupied p. 132–160 of Stevin's 2. *La Pratique D'Arithmetique*, also published at Leiden in 1585; there are copies of this work in the Plimpton Library of Columbia University, the Royal Library of Belgium, University of Liége, etc. A facsimile of this edition was published by G. SARTON in "The first explanation of decimal fractions and measures (1585). Together with a history of the decimal idea and a facsimile (no. XVII) of Stevin's Disme," *Isis*, v. 23, 1935, p. 230–244. A French edition, ed. by A. Girard, appeared also in a new edition of 3. *L'Arithmetique*, Leiden, 1625, p. 823–849, and in 4. S. STEVIN, *Les Oeuvres Mathematiques*, Leiden, 1634, p. 206–213. There were other Flemish editions, 5–6, in 1626 (in the Brown University Library), and in 1630.

This was the first work to set forth the theory of decimal fractions. It was translated into English by 7. ROBERT NORTON, "engineer and gunner" (d. 1635, *Dict. Nat. Biog.*), son of the poet THOMAS NORTON (1532–1584),

under the title *Disme: The Art of Tenths, or, Decimall Arithmetike,Teaching how to performe all Computations whatsoeuer, by whole Numbers without Fractions, by the foure Principles of Common Arithmeticke . . . Inuented by the excellent Mathematician Simon Steuin*, London, 1608; there are copies of this work in the British Museum, the Cambridge University Library, and in the Harvard University Library (a copy formerly in the W. A. White Library). According to the *Cambridge Bibliography of English Literature*, there were other editions, 8(?) in 1614 and, by HENRY LYTE, in 1619. The last statement is certainly incorrect in the case of Lyte's independent work (with a similar title), of which copies are in the Huntington Library, Pasadena, and the British Museum. See further SARTON, l.c., p. 158f. Norton supplied tables of interest and measurement, and instructions in decimal arithmetic to Robert Recorde's *The Grovnd of Artes*, London, 1623. See also H. Bosmans, "La 'Thiende' de Simon Stevin," *Rev. d. Questions Sci.*, s. 3, v. 27, 1920, p. 109–139.

The first of two paragraphs in "Article V, Des Compvtations Astronomiqves" of the "Appendice" of *La Disme* (1585), with its illustration of Stevin's decimal notation (by "commencemens" he referred to the integral part of the decimal) is as follows:

Aians les anciens Astronomes parti le circle en 360 degrez, ils voioient que les computations Astronomiques d'icelles, auec leurs partitions, estoient trop labourieuses, pourtant ils ont parti chasque degré en certaines parties, & les mesmes autrefois en autant, &c. à fin de pouuoir par ainsi tousiours operer par nombres entiers, en choissisans la soixantiesme progression, parce que 60 est nombre mesurable par plusieurs mesures entieres, à sçauoir 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, mais si l'on peut croire l'experience (ce que nous disons par toute reuerence de la venerable antiquité & esmeu auec l'vtilité commune) certes la soixantiesme progression n'estoit pas la plus commode, au moins entre celles qui consistoient potentiellement en la nature, ains la dixiesme qui est telle: Nous nommons les 360 degrez aussi *commencemens*, les denotans ainsi 360(0), & chascun degré ou 1(0) se diuisera en 10 parties egales, desquelles chascune sera 1(1), puis chasque 1(1) en 10(2), & ainsi des autres, comme le semblable est faict par plusieurs fois ci deuant.

[In 4 places near the end of this quotation we have, for typographical reasons, printed (0), (1), (2), instead of the original notation of the figures 0, 1, 2 surrounded by circles.]

R. C. A.

30. FIRST PUBLISHED COMPOUND INTEREST TABLES.—These were 1. Simon Stevin's *Tafelen van Interest, Midtsgaders De Constructie der seluer*, [Tables of interest together with their construction], Antwerp, 1582, 92 p. A French translation from the Dutch was published in 2. Stevin's *La Pratique D'Arithmetique*, Leiden, 1585, and an edition 3. by Girard in 1625 was reprinted in 4. Stevin's *Oeuvres Mathematiques*, Leyden, 1634, p. 185–206. A facsimile of the original work is given in 5. C. M. WALLER ZEPER, *De oudste intresttafels in Italië, Frankrijk en Nederland met een herdruk van Stevins "Tafelen van Interest,"* Diss. Leiden, Amsterdam, 1937.

Stevin tells us in his preface (dated Leiden July 16, 1582) that the inventor of these tables was JEAN TRENCHANT, who gave specimens of them

in his *L'Arithmetiqve de Ian Trenchant departie en troys livres, Ensemble vn petit discours des changes, auec L'art de calculer aux Getons*, Lyons, 1558. See G. SARTON, "JEAN TRENCHANT, French mathematician of the second half of the sixteenth century," *Isis*, v. 21, 1934, p. 207–209; and H. BOSMANS, Soc. Sci. de Bruxelles, *Annales*, v. 33, 1909, part 1, p. 184–192.

Since the description by A. DeMorgan of Stevin's Tables (*Supplement to the Penny Cyclopaedia*, v. 2, London, 1846, p. 604) contains an interesting suggestion we shall quote this: "it is pretty certain that these tables of compound interest *suggested decimal fractions*, the account of which speedily follows them. They are constructed as follows:—Ten millions being taken as the base (or *root*, as Stevinus calls it), and a rate, say five per cent., being chosen, the present value of ten millions due at the end of 1, 2, &c., up to 30 years, are put in a column, to the nearest integer. By their sides are sums of their values, which give the present values of the several annuities of ten million, as follows:

Table d'Interest de 5 pour 100.

1	9523810	9523810
2	9070295	18594105
3	8638376	27232481
4	8227025	35459506
.	. . . . .	. . . . .
.	. . . . .	. . . . .
30	2313774	153924494[1]

The rates are from 1 to 16 per cent., and also for 1 in 15, 1 in 16, &c., to 1 in 22;[2] or, as the French say, denier quinze, denier seize, &c. At the end is a direction to dispense, when convenient, with some of the last figures."

"There is thus a virtual use of decimal fractions preceding the formal one. The same thing happens in the tables of RICHARD WITT, [*Arithmeticall questions, touching the Buying or Exchange of Annuities* . . . , London, 1613] . . . which we believe to be the first English tables of compound interest, and the first English work (except the translation of the *Disme* of Stevinus) in which decimals were used."

R. C. A.

[1] Apparently De Morgan is here quoting from Stevin's *Oeuvres*. It should be 153724494 as in the 1582 and 1585 editions.—EDITOR.

[2] The table for 1 in 20 is not given since it would be the same as the 5% table.—EDITOR.

31. FIRST PUBLISHED MORTALITY TABLE.—At a time when the city of Breslau is so much in the war news it is appropriate here to recall that mortality records of this city were the basis of a paper, written by EDMOND HALLEY (1656–1742) over 250 years ago, and published in R. So. London, *Phil. Trans.*, 1693, no. 196, p. 596–610, and no. 198, p. 654–656. It is entitled "An estimate of the degrees of the mortality of mankind, drawn from curious tables of the births and funerals at the city of *Breslaw;* with an attempt to ascertain the price of *annuities* upon lives." Halley reprinted this in his *Miscellanea Curiosa*, London, v. 1, 1705; second ed. 1708; third ed., 1726, p. 280–301. This was also reprinted in Institute of Actuaries, *J.*, v. 18, 1874, p. 251f and 262f. Here we have the beginnings of statistical science and the first published mortality table, in the form since followed, to show the annual movement of a population, the problem of survivorship, the average

duration of one or more lives, and the money values depending thereon. See C. WALFORD, *Insurance Cyclopaedia*, London, v. 5, 1878, p. 616–618; v. 1, 1871, "Breslau table of mortality," p. 360–363; H. Böckh, "Halley als Statistiker. Zur Feier des zweihundertjährigen Bestehens von Halleys Sterblichkeitstafel," L'Institut International de Statistique, *Bull.*, v. 7, 1893, p. 1–24; A. LOEWY, *Versicherungslexikon*, ed. by A. Manes, third ed., Berlin 1930, cols. 742–744, 998–1000; E. J. FARREN, "On indirect methods of acquiring knowledge.—The method of history.—The first table of mortality," *Assurance Mag.*, v. 1, 1851, p. 40f.

Perhaps the inclusion of the following item is warranted, in not being too far removed from the main topic of the note. In *Math. Gazette*, v. 14, Dec. 1929, p. 574f, I published a Note on "A letter of DeMoivre and a theorem of Halley." The Note consisted mainly of a letter from De Moivre (1667–1754) to Halley which I copied from *The Mathematical Magazine and Philosophical Repository*, 1761 (see *Math. Gazette*, v. 14, p. 387, and v. 15, p. 215f) where it is stated that the letter, which is undated, was never before published. The theorem gives the rate of interest for an annuity, in a compact formula, $1 + b - \sqrt{b^2 - 2by}$ (for explanation of the notation, see the source). I asked, "Can any reader state where Halley's theorem is to be found or otherwise assist in determining an approximate date for the letter?" A long and very interesting reply from F. P. White, of St. John's College, Cambridge, appeared in the *Math. Gazette*, v. 15, Oct. 1930, p. 213f. The formula was first given by Halley in his chapter, "Of compound interest and annuities,"[1] in the first edition (1705) of Sherwin's *Mathematical Tables*. Hence De Moivre's letter, which was found among Halley's papers after his death in 1742, may have been written before 1706.

Halley was in 1703 the successor of Wallis as Savilian professor of geometry at Oxford, and in 1721 followed Flamsteed as astronomer royal. Due to his efforts the publication in 1687 of Newton's *Principia* was achieved. Among his contributions to astronomy was the discovery in 1682 of the comet, with a period of 76 years, last seen in 1910. He was also the translator from the Arabic and restorer of works of Apollonius of Perga (Oxford, 1706–1710).

R. C. A.

[1] It was reprinted in *Assurance Mag. and J.*, v. 9, 1861, p. 259–269.

32. GUIDE (*MTAC* no. 7), SUPPL. 1.—In F. BERGER, "Über den Temperaturverlauf in einem Zylinder von endlicher Länge beim Abkühlen und Erwärmen," *Z. f. angew. Math. u. Mech.*, v. 11, 1931, p. 53, there are 8 values (multiplied by 80), to 4–5D, of roots of the equation $xJ_1(x) = .53J_0(x)$. On p. 50 are graphical solutions of the first five roots. In GERALD PICKETT, "The effect of Biot's modulus on transient thermal stresses in concrete cylinders," Am. So. Testing Materials, *Trans.*, v. 39, 1939, p. 916, a table is given of the first six roots of

$$xJ_1(x) = BJ_0(x),$$

$B = .1, .5, 1, 3, 10, 100, 1000, 10\,000$. The first five roots are to 3D and the sixth to 2D.

In H. A. EINSTEIN, "Der Geschiebbetrieb als Wahrscheinlichkeits-

problem," diss. Eidgen. techn. Hochschule, Zürich, Versuchsanstalt für Wasserbau, *Mitteilung*, 1937, p. 93, there are graphs of the functions $e^{-x}I_0(x)$ and $e^{-x}I_1(x)$ on a logarithmic scale in which x ranges from .1 to about 100. T. II gives 3D for $e^{u}I_0(u)$, $u = 2(xT)^{\frac{1}{2}}$, and 4D for exp. $[-(x^{\frac{1}{2}} - T^{\frac{1}{2}})^2]e^{-u}I_0(u)$, $T = 0.895, 1.689, 3.14, 6.05, 11.48, 18.57$; $\sqrt{x}$ has various values from 0 to 6.18, a different set for each value of T.

In I. E. GARRICK, "On some reciprocal relations in the theory of nonstationary flows," U. S. Nat. adv. Comm. f. Aeronautics (NACA), *Techn. Reports*, no. 629, 1938, p. 348, for Theodorsen's function,

$$C(k) = H_1^{(2)}(k)/[H_1^{(2)}(k) + iH_0^{(2)}(k)] = F(k) + iG(k),$$

there are graphs of

$$F(k) = [J_1(J_1 + Y_0) + Y_1(Y_1 - J_0)]/[(J_1 + Y_0)^2 + (Y_1 - J_0)^2]$$

$$G(k) = -[Y_1Y_0 + J_1J_0]/[(J_1 + Y_0)^2 + (Y_1 - J_0)^2],$$

$0 \leqslant k \leqslant 2$, and also of $\sqrt{(F^2 + G^2)}$ and $\tan^{-1}(G/F)$, $0 \leqslant k \leqslant 1$. A table with 4D for $k = .1(.1).6(.2)1, 2, 4, 6, 10$ is given by T. THEODORSEN in the paper "General theory of aerodynamic instability and the mechanism of flutter," NACA, *Techn. Reports*, no. 496, 1934, p. 426, $J_0(k)$, $J_1(k)$, $Y_0(k)$, $Y_1(k)$, $F(k)$, and with 3–4D for F, and G, for the same values of k and also $k = 0, .025, .05, \infty$. On p. 418 there are graphs of F and G against $1/k$ for $0 \leqslant 1/k \leqslant 40$. On p. 350 Garrick gives a table with 4D of Wagner's function $k_1(x)$ for $x = 0(.5)1(1)10, 20, \infty$, and also a graph (p. 347), where $k_1(x) = (2/\pi)\int_0^\infty \sin(xt)F(t)dt/t = 1 + (2/\pi)\int_0^\infty \cos(xt)G(t)dt/t$. These values of $k_1(x)$ are taken from a paper by H. WAGNER, "Über die Entstehung des dynamischen Auftriebes von Tragflügeln," *Z. angew. Math. u. Mech.*, v. 5, 1925, p. 31.

In MAX JACOB, "Influence of nonuniform development of heat upon the temperature distribution in electrical coils and similar heat sources of simple form," A.S.M.E., *Trans.*, v. 65, 1943, p. 597, there is a table of the function

$$\frac{J_0(xy) - J_0(x)}{1 - J_0(x)} \qquad \begin{matrix} x = 0, .1, 1, 2, 2.3, 2.4048; \\ y = 0(.1)1. \end{matrix}$$

In Fig. 3 the values $x = 0$, 1, 2 and 2.4048 are used.

In C. WEBER, "Zum Zerfall eines Flüssigkeitsstrahles," *Z. f. angew. Math. u. Mech.*, v. 11, 1931, p. 150, values are given for $f_0(x)$, $f_1(x)$, where $f_0(x) = K_0(x)/K_1(x)$, and $f_1(x) = x/[xf_0(x) + 1]$, $x = [.1(.1)1(.2)2(.5)4(1)9;$ 3D].

H. B.

33. A NEW RESULT CONCERNING A MERSENNE NUMBER.—On 2 December 1944 I finished the proof that the Mersenne number M_{167} is composite; see *MTAC*, p. 333. Hence there now remain only four of these numbers M_p ($p = 193, 199, 227, 229$) whose character is unknown. The method of testing primality was the same as that described in Nat. Acad. Sci., *Proc.*, v. 30, Oct. 1944, p. 314–316.

H. S. UHLER

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34. ORIGINATORS OF THE TERM RADIAN.—As long ago as 1910 THOMAS MUIR pointed out (*Nature*, v. 83, p. 156) that while the earliest recorded use of the term *Radian* in the *New English Dictionary* was in 1879, in the first part of the new edition of the first volume of William Thomson and P. G. Tait's *Treatise on Natural Philosophy*, "my own first use of it was in class-teaching in the College Hall at St. Andrews in 1869, and I possess a notebook, belonging to one of my students of that year, in which the word is used." He hesitated, however, in a definite choice between the terms radial, radian, rad. But he states that as a result of reading a publication of A. J. ELLIS (1814–1890) and exchanging letters with him in 1874 "the form radian was definitely adopted by me." Ellis remarks that he had used the term "Radial angle" from his Cambridge undergraduate days, but Muir stated that Ellis approved of radian as a contraction of "*radi*-al *an*-gle." From later correspondence in *Nature*, v. 83, p. 217, 459–460, it appears that, wholly independent of Muir, JAMES THOMSON (1822–1892), brother of the above-mentioned William, proposed the name Radian in July 1871 and that he used it in an examination paper at Queen's College, Belfast, on June 5, 1873, published in the college calendar for 1873–74.

Bibliographic reports on the use before 1869 of the term *Radial Angle*, as equivalent to *Radian*, are desired. This term is not listed in *N.E.D.*

R. C. A.

35. *Phil. Mag.* TABLES, SUPPL. 3 (for Suppl. 1–2, see *MTAC*, p. 201–202).—W. G. BICKLEY, "Deflexions and vibrations of a circular elastic plate under tension," s. 7, v. 15, 1933, p. 795. The table gives, to 5S, the first two roots of

$$\frac{xJ_{n+1}(x)}{J_n(x)} = -\frac{\sqrt{(x^2+c^2)}I_{n+1}\sqrt{(x^2+c^2)}}{I_n\sqrt{(x^2+c^2)}}$$

for $n = 1$, $c = 0, 1, 2, 5, 10, 20$; the values of x and $x^2(x^2+c^2)$ are given, and for $n = 2$ the same quantities are given for the first root. This item was overlooked in the *Guide*, *MTAC* 7.

H. B.

36. ZEROS OF THE BESSEL FUNCTION $J_\nu(x)$.—If we denote, as usual, the k-th positive zero of $J_\nu(x)$ by $j_{\nu,k}$ then the symmetric function

$$\sigma_{2n}(\nu) = \sum_{k=1}^{\infty} (j_{\nu,k})^{-2n}$$

is, for each positive integer n, a rational function of ν. It was first used by RAYLEIGH[1] for the calculation of $j_{0,1}$ and $j_{1,1}$ and later by AIREY[2] and others for many values of $j_{\nu,1}$. These functions $\sigma_{2n}(\nu)$ are also important as coefficients of the meromorphic functions

$$\tfrac{1}{2}J_{\nu+1}(X)/J_\nu(X) = \sum_{n=1}^{\infty} \sigma_{2n}(\nu)X^{2n-1}$$

$$\tfrac{1}{2}J_\nu(X)/J_{\nu+1}(X) = (\nu+1)X^{-1} - \sum_{n=1}^{\infty} \sigma_{2n}(1+\nu)X^{2n-1}.$$

This last expansion, in effect, was given as far as $n = 4$ by JACOBI[3] in 1849. This is doubtless the first appearance of these functions σ. Later writers have given lists of these functions as follows:

author		range of n
RAYLEIGH[1]	(1874)	1(1)5, 8
GRAF & GUBLER[4]	(1898)	1(1)5
NIELSEN[5]	(1904)	1(1)5
KAPTEYN[6]	(1906)	1(1)6
FORSYTH[7]	(1920)	1(1)3
WATSON[8]	(1922)	1(1)5, 8

As a by-product of a recent investigation the first dozen of the functions were computed and are given below. They have interesting properties which may be discussed elsewhere. We need only the following explanations here. If we use $[x]$ to denote, as usual, the greatest integer $\leqslant x$, and if we define the polynomial $\pi_n(\nu)$ by

$$\pi_n(\nu) = \prod_{k=1}^{n} (k - \nu)^{[n/k]}$$

then the function $\phi_n(\nu)$ defined by

$$\sigma_{2n}(\nu) = 2^{-2n}\phi_n(\nu)/\pi_n(\nu)$$

is a polynomial of degree

$$d = 1 - n + \sum_{k=2}^{n} \left[\frac{n}{k}\right].$$

If we write

$$\phi_n(\nu) = a_0^{(n)} + a_1^{(n)}\nu + \cdots + a_d^{(n)}\nu^d,$$

then the coefficients $a_k^{(n)}$ for $n \leqslant 12$ are given in the following table.

k	$n=1$	2	3	4	5	6	7	8	9
0	1	1	2	11	38	946	4580	202738	3786092
1				5	14	1026	4324	311387	6425694
2						362	1316	185430	4434158
3						42	132	53752	1596148
4								7640	317136
5								429	33134
6									1430

k	$n=10$	11	12
0	261868876	1992367192	2381255244240
1	579783114	4028104212	7315072725560
2	547167306	3458238276	10093635442028
3	287834558	1649756012	8275251041478
4	92481350	479550668	4491836314618
5	18631334	87264812	1701744523728
6	2305702	9748732	461790600920
7	160850	614228	90534103564
8	4862	16796	12743301316
9			1257916872
10			82812812
11			3271518
12			58786

Thus for $n = 6$ we have

$$\sigma_{12}(\nu) = \frac{42\nu^3 + 362\nu^2 + 1026\nu + 946}{2^{12}(\nu+1)^6(\nu+2)^3(\nu+3)^2(\nu+4)(\nu+5)(\nu+6)}.$$

D. H. L.

[1] RAYLEIGH, London Math. So. *Proc.*, s. 1, v. 5, 1874, p. 119–124; *Scientific Papers*, v. 1, 1899, p. 192, 195. The entry for $n=8$ is due to A. CAYLEY; see also his *Collected Papers*, v. 9, 1896, p. 20.

[2] J. R. AIREY, *Phil. Mag.*, s. 6, v. 41, 1921, p. 200–203.

[3] C. G. J. JACOBI, *Astr. Nachrichten*, v. 28, 1849, cols. 93–94; *Gesammelte Werke*, Berlin, v. 7, 1891, p. 173 [*for* $10i + 32$, *read* $10i + 22$].

[4] J. H. GRAF & E. GUBLER, *Einleitung in die Theorie der Bessel'schen Funktionen*, v. 1, Bern, 1898, p. 130.

[5] N. NIELSEN, *Handbuch der Theorie der Cylinderfunktionen*, Leipzig, 1904, p. 360.

[6] W. KAPTEYN, *Archives Néerlandaises d. Sci. exactes et nat.*, s. 2, v. 11, 1906, p. 149, 168.

[7] A. R. FORSYTH, *Mess. Math.*, s. 2, v. 50, 1920, p. 135.

[8] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge, 1922 or 1944, p. 502.

QUERIES

13. TABLES OF INTEGRALS.—We are now interested in evaluating integrals of the following forms: $\int_x^\infty e^{-t}dt/t^n$, $\int_x^\infty e^{-t^2}dt/t^{2n}$. Are there published tables of these functions?

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EDITORIAL NOTE: Among many tables of $\int_x^\infty e^{-t}dt/t = -Ei(-x)$ reference may be given to NYMTP, *Tables of Sine, Cosine and Exponential Integrals*, 2v., 1940, for x = [0(.0001)1.9999; 9D], [0(.001)10; 9S], [10(.1)15; 14D]. There are useful Bibliographies in the volumes. When n is a positive integer $\int_x^\infty e^{-t}dt/t^n$ may be made to depend upon $Ei(-x)$. For the cases $n = +2(-1) - 2$ tables were published by W. L. Miller & T. R. Rosebrugh, R. So. Canada, *Proc. and Trans.*, series 2, section III, v. 9, 1903, p. 80–101, for x = [.1(.001)1(.01)2; 9D]. There are also tables (p. 80–81) of $-\int_x^\infty e^{-t}dt/t^2 + 1/x + \ln x$, and $-\int_x^\infty e^{-t}dt/t - \ln x$, for x = [0(.001).1; 9D]. In the case of the second integral, when $n = 0$ we have the error function of which the most extensive table is that of A. A. MARKOV, *Table des Valeurs de l'Intégrale* $\int_x^\infty e^{-t^2}dt$, St. Petersburg, 1888, for x = [0(.001)3(.01)4.8; 11D] with Δ^3; see *MTAC*, p. 136. However a more extensive table of the closely related function $H(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ has been published in NYMTP, *Tables of Probability Functions*, v. 1, 1941, x = [0(.0001)1(.001)5.6; 15D]. This table can be used to evaluate the above integral by means of the relation $\int_x^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}[1 - H(x)]$. Are there other tables of the first function than for $-2 > n > 2$, and of the second for $n \neq 0$?

QUERIES—REPLIES

14. TABLES OF $N^{3/2}$ (Q **5**, p. 131; QR **8**, p. 204; **11**, p. 336; **13**, p. 375).—We have ms. tables, to 10S, as follows for:

N = 100(1)1000, 1000(10)10 000, 1005(10)1565, and also
N = [1.0001(.0001)1.0099; 9D].

On setting $(1 + \alpha)^{3/2} = 1 + (3/2)\alpha + \phi$, we found that

for $0 \leqslant \alpha \leqslant 0.000\,036\,514$, $\phi = 0.000\,000\,000$;
for $0.000\,063\,246 \leqslant \alpha \leqslant 0.000\,081\,650$, $\phi = 0.000\,000\,002$;
for $0.000\,096\,610 \leqslant \alpha \leqslant 0.000\,109\,544$, $\phi = 0.000\,000\,004$.

IRWIN ROMAN

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CORRIGENDA

P. 168, l. 15, *for* 220, *read* 120.
P. 229, **IIIC** 4, *for* $iH_0^{(2)}(-ix)$, *read* $-iH_0^{(2)}(-ix)$; *for* $H_1^{(2)}(-ix)$, *read* $-H_1^{(2)}(-ix)$.
P. 241, **VIB** 10, *for* $-.4(.01)+1.49$, *read* $-.49(.01)+1.49$; *for* $.4(.01)+.99$, *read* $.49(.01).99$.
P. 265, l. 33, *for* $+e^{-z}w_n(-x)$, *read* $-e^{-z}w_n(-x)$.
P. 369, l. 11, *for* $e = 0(1^o)99^o$, *read* $e = \cos\theta$, $\theta = 0(1^o)99^o$.

CLASSIFICATION OF TABLES, AND SUBCOMMITTEES

A. Arithmetical Tables. Mathematical Constants
B. Powers
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D. Circular Functions
E. Hyperbolic and Exponential Functions
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The addresses of all contributors to each issue of *MTAC* are given in that issue, those of the Committee being on cover 2. The use of initials only indicates a member of the Executive Committee.

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